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ANALYSIS OF A TM_{01} CIRCULAR TO TE_{10} RECTANGULAR
WAVEGUIDE MODE CONVERTER

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By

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20. ABSTRACT (continued)

combination of expansion functions each of which is approximately the magnetic current associated with one of the rectangular waveguide modes. Substituting this \underline{M} into the equation of continuity of the tangential magnetic field across the apertures and testing, as in Galerkin's method, this equation with the expansion functions for \underline{M} , we arrive at a matrix equation which determines the coefficients of the expansion functions in the expansion for \underline{M} . In turn, \underline{M} determines the field in the waveguides, and, in particular, the amplitudes of the TE_{10} modes in the rectangular waveguides. Detailed formulas for the matrix elements are given in terms of Bessel functions and roots of Bessel functions. Further work is required to program these formulas in order to obtain numerical results using a digital computer.

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Chapter 1

Statement of the Problem

Two apertures in a circular waveguide feed two identical rectangular waveguides as shown in Fig. 1.1. The walls of each waveguide are perfectly conducting. The interiors of the left-hand waveguide, the right-hand waveguide, and the circular waveguide are called regions 1, 2, and 3, respectively. Homogeneous space of permeability μ and permittivity ϵ exists in regions 1, 2, and 3. The excitation is a TM_{01} wave of unit amplitude traveling in the z direction in the circular waveguide. The circular waveguide is of radius a and is terminated by a perfectly conducting wall at $z = L_3$. The radius a is such that only the TE_{11} and TM_{01} modes can propagate in the circular waveguide. A problem similar to the one being described was previously treated in [1].

Both rectangular waveguides run parallel to the x axis. Both have the same cross section $(-\frac{b}{2} \leq y \leq \frac{b}{2}, -\frac{c}{2} \leq z \leq \frac{c}{2})$ where $c < b$ and b is such that only the TE_{10} dominant mode can propagate in each rectangular waveguide. The aperture which feeds the left-hand rectangular waveguide in Fig. 1.1 is called A_1 . This aperture is the surface for which $(\rho = a, \pi - \phi_0 \leq \phi \leq \pi + \phi_0, -\frac{c}{2} \leq z \leq \frac{c}{2})$ where

$$\rho = \sqrt{x^2 + y^2} \quad (1.1)$$

$$\phi_0 = \sin^{-1}\left(\frac{b}{2a}\right) \quad (1.2)$$

The aperture which feeds the right-hand rectangular waveguide in Fig. 1.1 is called A_2 . This aperture is the surface for which $(\rho = a, -\phi_0 \leq \phi \leq \phi_0, -\frac{c}{2} \leq z \leq \frac{c}{2})$.

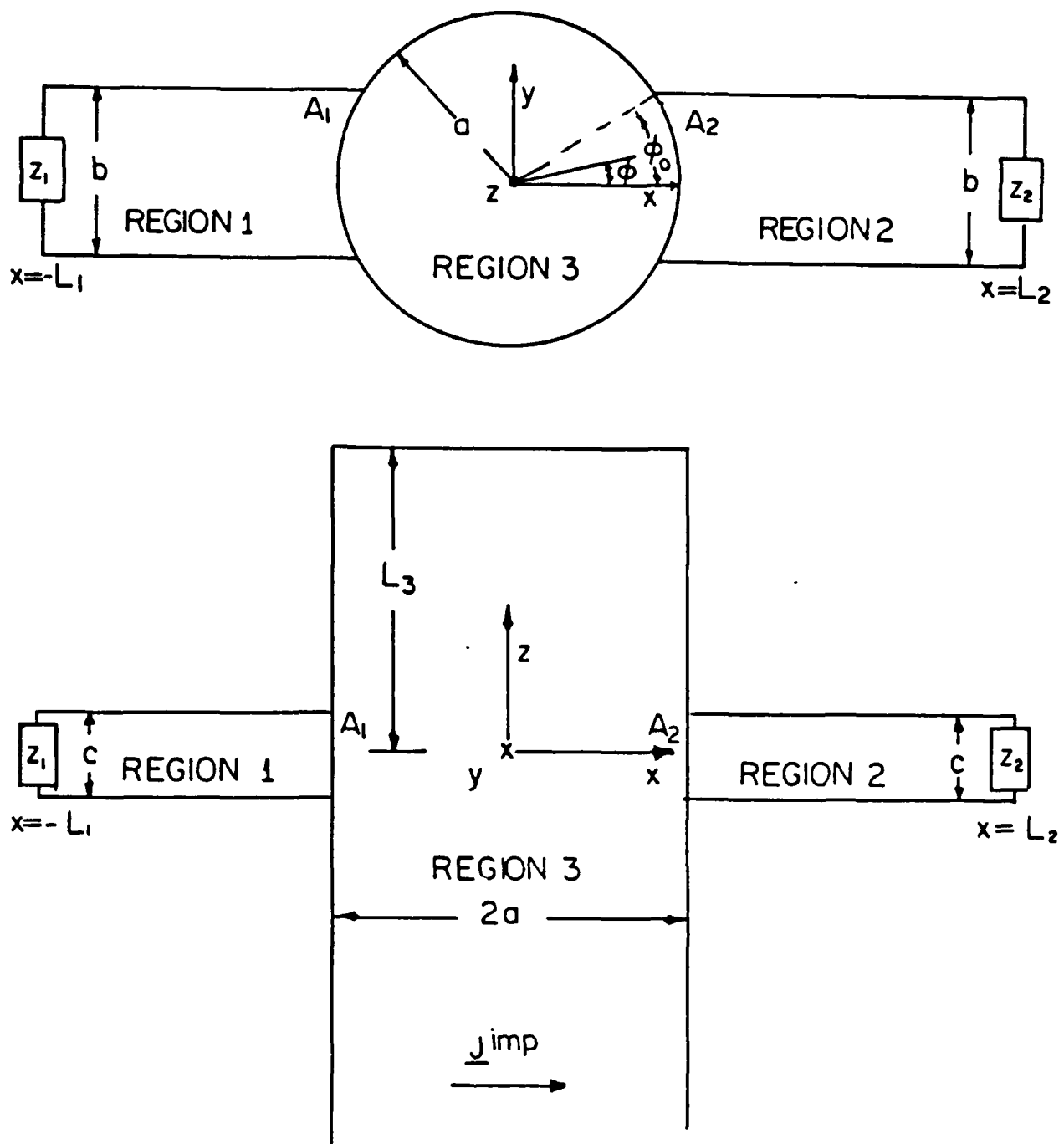


Figure 1.1: Top and side views of the TM_{01} to TE_{10} mode converter.

The voltage to current ratio of the TE_{10} mode in region 1 is taken to be Z_1 at $x = -L_1$. All other rectangular waveguide modes are evanescent. The voltage to current ratios of the evanescent modes at $x = -L_1$ do not come into play because L_1 is taken to be so large that any evanescent wave emanating from the termination at $x = -L_1$ will have negligible amplitude upon arrival at the aperture A_1 . The voltage to current ratio of the TE_{10} mode in region 2 is taken to be Z_2 at $x = L_2$. Here, L_2 is taken to be so large that any evanescent wave emanating from the termination at $x = L_2$ will have negligible amplitude upon arrival at the aperture A_2 .

As previously stated, the excitation is a z traveling (traveling in the z direction) TM_{01} wave of unit amplitude in the circular waveguide. By assumption, this is the only z traveling wave at $z = -\frac{\epsilon}{2}$. This wave is produced by an impressed source or sources located in the region for which $z \leq z_s < -\frac{\epsilon}{2}$ of the circular waveguide. Since neither of the rectangular waveguides nor the region for which $z > z_s$ of the circular waveguide contains an independent source, the electromagnetic field for $z > z_s$ in the circular waveguide depends only on the amplitudes of the z traveling waves at $z = z_s$, and not on the manner in which these amplitudes were produced. Therefore, we can, without loss of generality, assume that the impressed source of the unit amplitude z traveling TM_{01} wave in the circular waveguide is an electric current source \underline{J}^{imp} whose $-z$ traveling (traveling in the $-z$ direction) waves see a matched load, that is, these waves are never reflected. As shown in Fig. 1.1, \underline{J}^{imp} is located at $z < -\frac{\epsilon}{2}$ in the circular waveguide. The objective is to find the electromagnetic field in regions 1 and 2 of Fig. 1.1 and in the portion of region 3 for which $z > z_s$ in Fig. 1.1.

Chapter 2

Formulation

Following the generalized network formulation for aperture problems [2], [3], we close the apertures A_1 and A_2 with perfect electric conductors of infinitesimal thickness. As shown in Fig. 2.1, we place the surface density of magnetic current $\underline{M}^{(1)}$ on the region 1 side of the closed aperture A_1 , $-\underline{M}^{(1)}$ on the region 3 side of A_1 , $\underline{M}^{(2)}$ on the region 2 side of the closed aperture A_2 , and $-\underline{M}^{(2)}$ on the region 3 side of A_2 . In Fig. 1.1, the tangential electric field and the tangential magnetic field are continuous across A_1 and A_2 . The arrangement of magnetic currents in Fig. 2.1 ensures continuity of the tangential electric field across A_1 and A_2 . Now, the fields in Fig. 2.1 will be the same as those in Fig. 1.1 if $\underline{M}^{(1)}$ and $\underline{M}^{(2)}$ are adjusted so that the tangential magnetic field in Fig. 2.1 is continuous across A_1 and A_2 .

Continuity of the tangential magnetic field across A_1 in Fig. 2.1 is expressed as

$$\underline{H}_{\text{tan}}^{(1)} = \underline{H}_{\text{tan}}^{(3)} \quad \text{on } A_1 \quad (2.1)$$

where $\underline{H}^{(1)}$ is the magnetic field in region 1 and $\underline{H}^{(3)}$ is the magnetic field in region 3. In (2.1), the subscript "tan" denotes the components tangent to A_1 . Continuity of the tangential magnetic field across A_2 in Fig. 2.1 is expressed as

$$\underline{H}_{\text{tan}}^{(2)} = \underline{H}_{\text{tan}}^{(3)} \quad \text{on } A_2 \quad (2.2)$$

where $\underline{H}^{(2)}$ is the magnetic field in region 2. In (2.2), the subscript "tan" denotes the components tangent to A_2 .

The electromagnetic field ($\underline{E}^{(1)}, \underline{H}^{(1)}$) in region 1 of Fig. 2.1 is due to $\underline{M}^{(1)}$

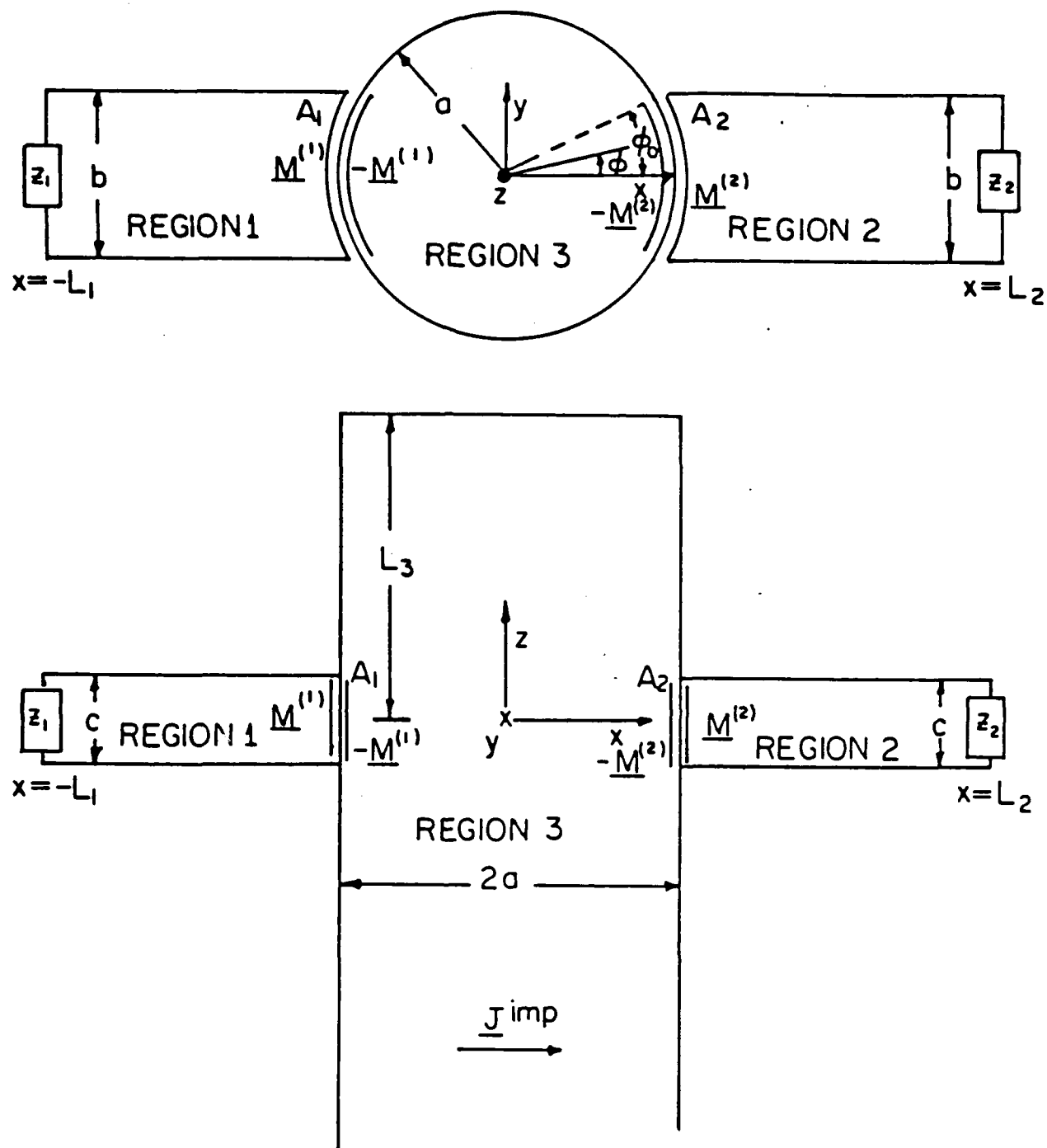


Figure 2.1: Top and side views of the situation equivalent to that of Fig. 1.1.

placed on the region 1 side of the closed aperture A_1 :

$$\underline{E}^{(1)} = \underline{E}^{(1)}(\underline{Q}, \underline{M}^{(1)}) \quad (2.3)$$

$$\underline{H}^{(1)} = \underline{H}^{(1)}(\underline{Q}, \underline{M}^{(1)}) \quad (2.4)$$

The superscript "(1)" on \underline{E} and \underline{H} on the right-hand sides of (2.3) and (2.4) denotes radiation in region 1 of Fig. 2.1. The " \underline{Q} " on the right-hand sides of (2.3) and (2.4) indicates that there is no electric current source.

The field ($\underline{E}^{(2)}, \underline{H}^{(2)}$) in region 2 of Fig. 2.1 is due to $\underline{M}^{(2)}$ placed on the region 2 side of the closed aperture A_2 :

$$\underline{E}^{(2)} = \underline{E}^{(2)}(\underline{Q}, \underline{M}^{(2)}) \quad (2.5)$$

$$\underline{H}^{(2)} = \underline{H}^{(2)}(\underline{Q}, \underline{M}^{(2)}) \quad (2.6)$$

The superscript "(2)" on \underline{E} and \underline{H} on the right-hand sides of (2.5) and (2.6) denotes radiation in region 2 of Fig. 2.1. The " \underline{Q} " on the right-hand sides of (2.5) and (2.6) indicates that there is no electric current source.

The field ($\underline{E}^{(3)}, \underline{H}^{(3)}$) for $z > z_L$ in region 3 of Fig. 2.1 is due to the current sources $\underline{J}^{\text{imp}}$, $-\underline{M}^{(1)}$, and $-\underline{M}^{(2)}$ radiating in the circular waveguide with the apertures A_1 and A_2 closed by perfect conductors, with the perfectly conducting wall at $z = L_3$, and with a matched load at $z = z_L$ where z_L is any value such that all of $\underline{J}^{\text{imp}}$ lies in the region for which $z > z_L$:

$$\underline{E}^{(3)} = \underline{E}^{(3)}(\underline{J}^{\text{imp}}, \underline{Q}) - \underline{E}^{(3)}(\underline{Q}, \underline{M}^{(1)}) - \underline{E}^{(3)}(\underline{Q}, \underline{M}^{(2)}) \quad (2.7)$$

$$\underline{H}^{(3)} = \underline{H}^{(3)}(\underline{J}^{\text{imp}}, \underline{Q}) - \underline{H}^{(3)}(\underline{Q}, \underline{M}^{(1)}) - \underline{H}^{(3)}(\underline{Q}, \underline{M}^{(2)}) \quad (2.8)$$

The superscript "(3)" on the right-hand sides of (2.7) and (2.8) denotes radiation in region 3 with the apertures closed, with the short at $z = L_3$, and with the matched load at the other end. The first argument of each field on the right-hand sides of (2.7) and (2.8) is an electric current source; the second argument is a magnetic current source.

Substitution of (2.4) and (2.8) into (2.1) gives

$$-\underline{H}_{\text{tan}}^{(1)}(\underline{Q}, \underline{M}^{(1)}) - \underline{H}_{\text{tan}}^{(3)}(\underline{Q}, \underline{M}^{(1)}) - \underline{H}_{\text{tan}}^{(3)}(\underline{Q}, \underline{M}^{(2)}) = -\underline{H}_{\text{tan}}^{(3)}(\underline{J}^{\text{imp}}, \underline{Q}) \quad \text{on } A_1 \quad (2.9)$$

Substitution of (2.6) and (2.8) into (2.2) yields

$$-\underline{H}_{\text{tan}}^{(3)}(\underline{Q}, \underline{M}^{(1)}) - \underline{H}_{\text{tan}}^{(2)}(\underline{Q}, \underline{M}^{(2)}) - \underline{H}_{\text{tan}}^{(3)}(\underline{Q}, \underline{M}^{(2)}) = -\underline{H}_{\text{tan}}^{(3)}(\underline{J}^{\text{imp}}, \underline{Q}) \quad \text{on } A_2 \quad (2.10)$$

Let

$$\underline{M}^{(1)} = \sum_{p=1}^{PTM} \sum_{q=1}^{QTM} V_{pq}^{1TM} \underline{M}_{pq}^{1TM}(\phi, z) + \sum_{p=0}^{PTE} \sum_{\substack{q=0 \\ p+q \neq 0}}^{QTE} V_{pq}^{1TE} \underline{M}_{pq}^{1TE}(\phi, z) \quad (2.11)$$

$$\underline{M}^{(2)} = \sum_{p=1}^{PTM} \sum_{q=1}^{QTM} V_{pq}^{2TM} \underline{M}_{pq}^{2TM}(\phi, z) + \sum_{p=0}^{PTE} \sum_{\substack{q=0 \\ p+q \neq 0}}^{QTE} V_{pq}^{2TE} \underline{M}_{pq}^{2TE}(\phi, z) \quad (2.12)$$

where the V 's are unknown coefficients to be determined and

$$\underline{M}_{pq}^{1\delta}(\phi, z) = \underline{u}_\phi e_{zpq}^\delta(y^{1+}, z^+) + \underline{u}_z \frac{\sin \phi_o}{\phi_o} e_{ypq}^\delta(y^{1+}, z^+), \quad \left\{ \begin{array}{l} \rho = a \\ \pi - \phi_o \leq \phi \leq \pi + \phi_o \\ -\frac{\epsilon}{2} \leq z \leq \frac{\epsilon}{2} \end{array} \right. \quad (2.13)$$

$$\underline{M}_{pq}^{2\delta}(\phi, z) = \underline{u}_\phi e_{zpq}^\delta(y^{2+}, z^+) - \underline{u}_z \frac{\sin \phi_o}{\phi_o} e_{ypq}^\delta(y^{2+}, z^+), \quad \left\{ \begin{array}{l} \rho = a \\ -\phi_o \leq \phi \leq \phi_o \\ -\frac{\epsilon}{2} \leq z \leq \frac{\epsilon}{2} \end{array} \right. \quad (2.14)$$

where \underline{u}_ϕ and \underline{u}_z are the unit vectors in the ϕ and z directions, respectively. Furthermore, e_{ypq}^δ and e_{zpq}^δ are, respectively, the y and z components of $\underline{e}_{pq}^\delta$. Now, δ is either TM or TE , and \underline{e}_{pq}^{TM} and \underline{e}_{pq}^{TE} are given by (A.10) and (A.23), respectively. In (2.13) and (2.14),

$$y^{1+} = (\pi - \phi)x_o + \frac{\epsilon}{2} \quad (2.15)$$

$$y^{2+} = \phi x_o + \frac{\epsilon}{2} \quad (2.16)$$

$$z^+ = z + \frac{\epsilon}{2} \quad (2.17)$$

where

$$x_o = \frac{a \sin \phi_o}{\phi_o} \quad (2.18)$$

In Chapter 3, it is shown that $(\underline{E}^{(1)}(Q, \underline{M}_{pq}^{1TM}), \underline{H}^{(1)}(Q, \underline{M}_{pq}^{1TM}))$ is approximately a TM_{pq} field in region 1, that $(\underline{E}^{(1)}(Q, \underline{M}_{pq}^{1TE}), \underline{H}^{(1)}(Q, \underline{M}_{pq}^{1TE}))$ is approximately a TE_{pq} field in region 1, that $(\underline{E}^{(2)}(Q, \underline{M}_{pq}^{2TM}), \underline{H}^{(2)}(Q, \underline{M}_{pq}^{2TM}))$ is approximately a TM_{pq} field in region 2, and that $(\underline{E}^{(2)}(Q, \underline{M}_{pq}^{2TE}), \underline{H}^{(2)}(Q, \underline{M}_{pq}^{2TE}))$ is approximately a TE_{pq} field in region 2.

Substituting (2.11) and (2.12) into (2.9) and (2.10), we obtain

$$\begin{aligned}
& - \sum_{p=1}^{PTM} \sum_{q=1}^{QTM} V_{pq}^{1TM} (H_{\tan}^{(1)}(\underline{Q}, \underline{M}_{pq}^{1TM}) + H_{\tan}^{(3)}(\underline{Q}, \underline{M}_{pq}^{1TM})) \\
& - \sum_{p=0}^{PTE} \sum_{q=0}^{QTE} V_{pq}^{1TE} (H_{\tan}^{(1)}(\underline{Q}, \underline{M}_{pq}^{1TE}) + H_{\tan}^{(3)}(\underline{Q}, \underline{M}_{pq}^{1TE})) \\
& \quad p+q \neq 0 \\
& - \sum_{p=1}^{PTM} \sum_{q=1}^{QTM} V_{pq}^{2TM} H_{\tan}^{(3)}(\underline{Q}, \underline{M}_{pq}^{2TM}) - \sum_{p=0}^{PTE} \sum_{q=0}^{QTE} V_{pq}^{2TE} H_{\tan}^{(3)}(\underline{Q}, \underline{M}_{pq}^{2TE}) \\
& \quad p+q \neq 0 \\
& = -H_{\tan}^{(3)}(J^{\text{imp}}, \underline{Q}) \quad \text{on } A_1 \quad (2.19)
\end{aligned}$$

and

$$\begin{aligned}
& - \sum_{p=1}^{PTM} \sum_{q=1}^{QTM} V_{pq}^{1TM} H_{\tan}^{(3)}(\underline{Q}, \underline{M}_{pq}^{1TM}) - \sum_{p=0}^{PTE} \sum_{q=0}^{QTE} V_{pq}^{1TE} H_{\tan}^{(3)}(\underline{Q}, \underline{M}_{pq}^{1TE}) \\
& \quad p+q \neq 0 \\
& - \sum_{p=1}^{PTM} \sum_{q=1}^{QTM} V_{pq}^{2TM} (H_{\tan}^{(2)}(\underline{Q}, \underline{M}_{pq}^{2TM}) + H_{\tan}^{(3)}(\underline{Q}, \underline{M}_{pq}^{2TM})) \\
& - \sum_{p=0}^{PTE} \sum_{q=0}^{QTE} V_{pq}^{2TE} (H_{\tan}^{(2)}(\underline{Q}, \underline{M}_{pq}^{2TE}) + H_{\tan}^{(3)}(\underline{Q}, \underline{M}_{pq}^{2TE})) \\
& \quad p+q \neq 0 \\
& = -H_{\tan}^{(3)}(J^{\text{imp}}, \underline{Q}) \quad \text{on } A_2 \quad (2.20)
\end{aligned}$$

The symmetric product $\langle A, B \rangle$ between two vectors \underline{A} and \underline{B} is defined to be the surface integral of their dot product over whichever aperture they are defined:

$$\langle A, B \rangle = \iint_{A_1 \text{ or } A_2} \underline{A} \cdot \underline{B} ds \quad (2.21)$$

Here, ds is the differential element of surface area. First taking the symmetric product of (2.19) with each of the expansion functions \underline{M}_{mn}^{1TM} and \underline{M}_{mn}^{1TE} that appear in (2.11), and then taking the symmetric product of (2.20) with each

of the expansion functions \underline{M}_{mn}^{2TM} and \underline{M}_{mn}^{2TE} that appear in (2.12), we obtain

$$[Y^1 + Y^2 + Y^3] \begin{bmatrix} \vec{V}^{1TM} \\ \vec{V}^{1TE} \\ \vec{V}^{2TM} \\ \vec{V}^{2TE} \end{bmatrix} = \begin{bmatrix} \vec{I}^{1TM} \\ \vec{I}^{1TE} \\ \vec{I}^{2TM} \\ \vec{I}^{2TE} \end{bmatrix} \quad (2.22)$$

where the \vec{V} 's are column vectors of the unknown V 's in (2.11) and (2.12). The \vec{I} 's are also column vectors; the right-hand side of (2.22) is called the excitation vector.

The elements of the \vec{V} 's in (2.22) are given by

$$V_j^{\gamma\delta} = V_{pq}^{\gamma\delta}, \quad \begin{cases} j = 1, 2, \dots, N^\delta \\ \gamma = 1, 2 \\ \delta = TM, TE \end{cases} \quad (2.23)$$

The subscript " j " on the left-hand side of (2.23) is a condensation of the double subscript " pq " on the right-hand side of (2.23); a one to one correspondence is established between each pair of integers p and q in use and each of the positive integers $1, 2, \dots, N^\delta$. The correspondence in the " TE " expressions ($\delta = TE$ in (2.23)) may be different from that in the " TM " expressions ($\delta = TM$ in (2.23)). The elements of the I 's in (2.22) are given by

$$I_i^{\alpha\beta} = - \iint_{A_\alpha} \underline{M}_{mn}^{\alpha\beta} \cdot \underline{H}^{(3)}(\underline{J}^{\text{imp}}, \underline{Q}) ds, \quad \begin{cases} i = 1, 2, \dots, N^\beta \\ \alpha = 1, 2 \\ \beta = TM, TE \end{cases} \quad (2.24)$$

where i is related to mn in the same way as j is related to pq in (2.23). The relationship in (2.23) was of the type " δ "; that in (2.24) is of the type " β ". The subscript "tan" which was attached to $\underline{H}^{(3)}(\underline{J}^{\text{imp}}, \underline{Q})$ in (2.19) is not needed in (2.24) because $\underline{M}_{mn}^{\alpha\beta}$ is tangent to A_α .

In (2.22), Y^1 , Y^2 , and Y^3 are the admittance matrices for regions 1, 2, and 3, respectively. The matrix Y^1 is given by

$$Y^1 = \begin{bmatrix} Y^{1,1TM,1TM} & Y^{1,1TM,1TE} & 0 & 0 \\ Y^{1,1TE,1TM} & Y^{1,1TE,1TE} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (2.25)$$

where each entry of the array on the right-hand side is a submatrix. Each element of the "0" submatrices is zero. The elements of the non-zero submatrices are given by

$$Y_{ij}^{1,1\beta,1\delta} = - \iint_{A_1} \underline{M}_{mn}^{1\beta} \cdot \underline{H}^{(1)}(\underline{Q}, \underline{M}_{pq}^{1\delta}) ds, \quad \begin{cases} i = 1, 2, \dots, N^\beta \\ j = 1, 2, \dots, N^\delta \\ \beta = TM, TE \\ \delta = TM, TE \end{cases} \quad (2.26)$$

where j is related to pq as in (2.23); i is similarly related to mn . Nevertheless, the relationship between i and mn may be different from that between j and pq . The relationship between i and mn is of the type " β ". The relationship between j and pq is of the type " δ ". The matrix Y^2 is given by

$$Y^2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & Y^{2,2TM,2TM} & Y^{2,2TM,2TE} \\ 0 & 0 & Y^{2,2TE,2TM} & Y^{2,2TE,2TE} \end{bmatrix} \quad (2.27)$$

where the elements of the "0" submatrices are zero. The elements of the non-zero submatrices are given by

$$Y_{ij}^{2,2\beta,2\delta} = - \iint_{A_2} \underline{M}_{mn}^{2\beta} \cdot \underline{H}^{(2)}(\underline{Q}, \underline{M}_{pq}^{2\delta}) ds, \quad \begin{cases} i = 1, 2, \dots, N^\beta \\ j = 1, 2, \dots, N^\delta \\ \beta = TM, TE \\ \delta = TM, TE \end{cases} \quad (2.28)$$

where i is related to mn and j is related to pq as in (2.26). The matrix Y^3 is given by

$$Y^3 = \begin{bmatrix} Y^{3,1TM,1TM} & Y^{3,1TM,1TE} & Y^{3,1TM,2TM} & Y^{3,1TM,2TE} \\ Y^{3,1TE,1TM} & Y^{3,1TE,1TE} & Y^{3,1TE,2TM} & Y^{3,1TE,2TE} \\ Y^{3,2TM,1TM} & Y^{3,2TM,1TE} & Y^{3,2TM,2TM} & Y^{3,2TM,2TE} \\ Y^{3,2TE,1TM} & Y^{3,2TE,1TE} & Y^{3,2TE,2TM} & Y^{3,2TE,2TE} \end{bmatrix} \quad (2.29)$$

where

$$Y_{ij}^{3,\alpha\beta,\gamma\delta} = - \iint_{A_\alpha} \underline{M}_{mn}^{\alpha\beta} \cdot \underline{H}^{(3)}(\underline{Q}, \underline{M}_{pq}^{\gamma\delta}) ds, \quad \begin{cases} i = 1, 2, \dots, N^\beta \\ j = 1, 2, \dots, N^\delta \\ \alpha = 1, 2 \\ \beta = TE, TM \\ \gamma = 1, 2 \\ \delta = TE, TM \end{cases} \quad (2.30)$$

In (2.30), i is related to mn and j is related to pq as in (2.26).

If the elements of the admittance matrices Y^1 , Y^2 , and Y^3 and the column vectors \vec{I}^{1TM} , \vec{I}^{1TE} , \vec{I}^{2TM} , and \vec{I}^{2TE} can be evaluated, then (2.22) can be solved for \vec{V}^{1TM} , \vec{V}^{1TE} , \vec{V}^{2TM} , and \vec{V}^{2TE} . These \vec{V} 's determine $\underline{M}^{(1)}$ and $\underline{M}^{(2)}$ according to (2.11) and (2.12). Next, $\underline{M}^{(1)}$ and $\underline{M}^{(2)}$ can be substituted into expressions (2.3)–(2.8) for the fields in the waveguide regions.

Chapter 3

The Admittance Matrices for the Rectangular Waveguides

In this chapter, $Y_{ij}^{1,1\beta,1\delta}$ of (2.26) and $Y_{ij}^{2,2\beta,2\delta}$ of (2.28) are evaluated. We approximate $Y_{ij}^{1,1\beta,1\delta}$ of (2.26) by $\hat{Y}_{ij}^{1,1\beta,1\delta}$ given by

$$\hat{Y}_{ij}^{1,1\beta,1\delta} = - \iint_{\hat{A}_1} \hat{\underline{M}}_{mn}^{1\beta} \cdot \underline{H}^{(1)}(\underline{Q}, \hat{\underline{M}}_{pq}^{1\delta}) ds, \begin{cases} i = 1, 2, \dots, N^\beta \\ j = 1, 2, \dots, N^\delta \\ \beta = TM, TE \\ \delta = TM, TE \end{cases} \quad (3.1)$$

We approximate $Y_{ij}^{2,2\beta,2\delta}$ of (2.28) by $\hat{Y}_{ij}^{2,2\beta,2\delta}$ given by

$$\hat{Y}_{ij}^{2,2\beta,2\delta} = - \iint_{\hat{A}_2} \hat{\underline{M}}_{mn}^{2\beta} \cdot \underline{H}^{(2)}(\underline{Q}, \hat{\underline{M}}_{pq}^{2\delta}) ds, \begin{cases} i = 1, 2, \dots, N^\beta \\ j = 1, 2, \dots, N^\delta \\ \beta = TM, TE \\ \delta = TM, TE \end{cases} \quad (3.2)$$

In (3.1) and (3.2), $\hat{\underline{M}}_{pq}^{1\delta}$ and $\hat{\underline{M}}_{pq}^{2\delta}$ are approximations to $\underline{M}_{pq}^{1\delta}$ of (2.13) and $\underline{M}_{pq}^{2\delta}$ of (2.14). Furthermore, \hat{A}_1 and \hat{A}_2 are, respectively, the surfaces on which $\hat{\underline{M}}_{pq}^{1\delta}$ and $\hat{\underline{M}}_{pq}^{2\delta}$ are defined. In (3.1) and (3.2), j is related to pq as in (2.23); i is similarly related to mn .

The Y 's of (3.1) and (3.2) will be evaluated by first defining the $\hat{\underline{M}}$'s that appear in (3.1) and (3.2) and then by finding the \underline{H} 's that appear in (3.1)

and (3.2). The electric fields associated with these \underline{H} 's will also be found because these electric fields are needed in order to evaluate, as indicated in the last two sentences of Chapter 2, the field in the rectangular waveguides. Finally, the $\hat{\underline{M}}$'s and the \underline{H} 's will be substituted into (3.1) and (3.2) to obtain appropriate expressions for the Y 's.

We define $\hat{\underline{M}}_{pq}^{1\delta}$ by

$$\hat{\underline{M}}_{pq}^{1\delta} = \hat{\underline{M}}_{pq}^{1\delta}(y, z) = -\underline{e}_{pq}^{\delta}(y^+, z^+) \times \underline{u}_x, \begin{cases} x = -x_o \\ -\frac{b}{2} \leq y \leq \frac{b}{2} \\ -\frac{c}{2} \leq z \leq \frac{c}{2} \end{cases} \quad (3.3)$$

where z^+ and x_o are given by (2.17) and (2.18), respectively. Furthermore,

$$y^+ = y + \frac{b}{2} \quad (3.4)$$

As defined by (3.3), $\hat{\underline{M}}_{pq}^{1\delta}$ exists on a portion of the $x = -x_o$ plane. With x_o given by (2.18), $-x_o$ is the average value of x over the curved surface on which $\underline{M}_{pq}^{1\delta}$ of (2.13) exists. $\hat{\underline{M}}_{pq}^{1\delta}$ is placed on the region 1 side of a perfect conductor which covers the plane surface in (3.3). This surface is called \hat{A}_1 .

We relate y of (3.4) to ϕ by

$$y = (\pi - \phi)x_o \quad (3.5)$$

Relationship (3.5) is reasonable because:

- 1) $y = 0$ when $\phi = \pi$
- 2) $y = \frac{b}{2}$ when $\phi = \pi - \phi_o$
- 3) $y = -\frac{b}{2}$ when $\phi = \pi + \phi_o$

The above items 2) and 3) are obtained by using (2.18) and (1.2). Given (3.5), the interval $(y, y + dy)$ corresponds to the interval $(\phi, \phi + d\phi)$ where ϕ is related to y by (3.5) and

$$d\phi = -\frac{dy}{x_o} \quad (3.6)$$

Here, dy and $d\phi$ are the differentials of y and ϕ , respectively. $\hat{\underline{M}}_{pq}^{1\delta}$ of (3.3) is a good approximation to $\underline{M}_{pq}^{1\delta}$ of (2.13) because the transverse and longitudinal

voltages produced by $\hat{M}_{pq}^{1\delta}$ on the portion of its surface for y in the interval $(y + dy)$ and z in the interval $(z + dz)$ are the same as the transverse and longitudinal voltages produced by $\underline{M}_{pq}^{1\delta}$ on the corresponding portion of its surface. The latter portion of surface is the surface for which $\rho = a$, ϕ is in the interval $(\phi, \phi + d\phi)$, and z is in the interval $(z, z + dz)$. Here, dz is the differential of z . Moreover, ϕ and $d\phi$ are related to y and dy by (3.5) and (3.6), respectively.

To show equality of the voltages mentioned in the previous paragraph, we note that $\hat{M}_{pq}^{1\delta}$ of (3.3) produces the tangential electric field $\hat{e}_{pq}^{1\delta}$ given by

$$\hat{e}_{pq}^{1\delta} = -\underline{u}_x \times \hat{M}_{pq}^{1\delta} = \underline{e}_{pq}^{\delta}(y^+, z^+) \quad (3.7)$$

on its surface and that $\underline{M}_{pq}^{1\delta}$ of (2.13) produces the tangential electric field $\underline{e}_{pq}^{1\delta}$ given by

$$\underline{e}_{pq}^{1\delta} = \underline{u}_\rho \times \underline{M}_{pq}^{1\delta} = -\underline{u}_\phi \frac{\sin \phi_o}{\phi_o} \underline{e}_{ypq}^{\delta}(y^{1+}, z^+) + \underline{u}_z \underline{e}_{zpq}^{\delta}(y^{1+}, z^+) \quad (3.8)$$

on its surface. Here, \underline{u}_x , \underline{u}_y , \underline{u}_z , \underline{u}_ρ , and \underline{u}_ϕ are unit vectors in the directions indicated by the subscripts. The transverse voltage produced by $\hat{M}_{pq}^{1\delta}$ is the voltage at $y + dy$ with respect to that at y . This voltage is \hat{V}_t given, in view of (3.7), by

$$\hat{V}_t = \hat{e}_{pq}^{1\delta} \cdot (-\underline{u}_y dy) = -\underline{e}_{ypq}^{\delta}(y^+, z^+) dy \quad (3.9)$$

The transverse voltage produced by $\underline{M}_{pq}^{1\delta}$ is the voltage at $\phi + d\phi$ with respect to that at ϕ . This voltage is V_t given by

$$V_t = \underline{e}_{pq}^{1\delta} \cdot (-\underline{u}_\phi a d\phi) = -\underline{e}_{ypq}^{\delta} \cdot (y^{1+}, z^+) dy \quad (3.10)$$

Equations (2.18), (3.6) and (3.8) were used to obtain (3.10). The longitudinal voltage produced by $\hat{M}_{pq}^{1\delta}$ is \hat{V}_l given by

$$\hat{V}_l = \hat{e}_{pq}^{1\delta} \cdot (-\underline{u}_z dz) = -\underline{e}_{zpq}^{\delta}(y^+, z^+) dz \quad (3.11)$$

The longitudinal voltage produced by $\underline{M}_{pq}^{1\delta}$ is V_l given by

$$V_l = \underline{e}_{pq}^{1\delta} \cdot (-\underline{u}_z dz) = -\underline{e}_{zpq}^{\delta}(y^{1+}, z^+) dz \quad (3.12)$$

From (2.15), (3.4), and (3.5), we have $y^{1+} = y^+$ so that it is now evident from (3.9)–(3.12) that the transverse and longitudinal voltages produced by $\hat{M}_{pq}^{1\delta}$ are the same as those produced by $M_{pq}^{1\delta}$.

We define $\hat{M}_{pq}^{2\delta}$ by

$$\hat{M}_{pq}^{2\delta} = \hat{M}_{pq}^{2\delta}(y, z) = e_{pq}^{\delta}(y^+, z^+) \times \underline{u}_x, \begin{cases} x = x_0 \\ -\frac{b}{2} \leq y \leq \frac{b}{2} \\ -\frac{c}{2} \leq z \leq \frac{c}{2} \end{cases} \quad (3.13)$$

where z^+ , x_0 , and y^+ are given by (2.17), (2.18), and (3.4), respectively. $\hat{M}_{pq}^{2\delta}$ of (3.13) is a good approximation to $M_{pq}^{2\delta}$ of (2.14) because the transverse and longitudinal voltages produced by $\hat{M}_{pq}^{2\delta}$ on a differential element of its surface are the same as the transverse and longitudinal voltages produced by $M_{pq}^{2\delta}$ on the corresponding differential element of its surface. Equality of these voltages follows from an argument similar to that in the previous paragraph.

The problem of finding the field $(\underline{E}^{(1)}(\underline{Q}, \hat{M}_{pq}^{1\delta}), \underline{H}^{(1)}(\underline{Q}, \hat{M}_{pq}^{1\delta}))$ is a boundary value problem in which the transverse electric field at $x = -x_0$ in region 1 is $\hat{e}_{pq}^{1\delta}$ given by (3.7) as

$$\hat{e}_{pq}^{1\delta} = e_{pq}^{\delta}(y^+, z^+) \quad (3.14)$$

Furthermore, region 1 is terminated, as in Fig. 2.1, at $x = -L_1$ with the impedance Z_1 . Therefore, as explained in the third paragraph of Chapter 1, the only x traveling wave at $x = -x_0$ is a TE_{10} wave.

We choose

$$(\underline{E}^{(1)}(\underline{Q}, \hat{M}_{pq}^{1TM}), \underline{H}^{(1)}(\underline{Q}, \hat{M}_{pq}^{1TM})) = C_{pq}^{TM-}(\underline{E}_{pq}^{TM-}, \underline{H}_{pq}^{TM-}) \quad (3.15)$$

$$\begin{aligned} (\underline{E}^{(1)}(\underline{Q}, \hat{M}_{10}^{1TE}), \underline{H}^{(1)}(\underline{Q}, \hat{M}_{10}^{1TE})) &= C_{10}^{TE-}(\underline{E}_{10}^{TE-}, \underline{H}_{10}^{TE-}) \\ &\quad + C_{10}^{TE+}(\underline{E}_{10}^{TE+}, \underline{H}_{10}^{TE+}) \end{aligned} \quad (3.16)$$

$$(\underline{E}^{(1)}(\underline{Q}, \hat{M}_{pq}^{1TE}), \underline{H}^{(1)}(\underline{Q}, \hat{M}_{pq}^{1TE})) = C_{pq}^{TE-}(\underline{E}_{pq}^{TE-}, \underline{H}_{pq}^{TE-}), (p, q \neq 1, 0) \quad (3.17)$$

where the C 's are unknown constants. The $(\underline{E}, \underline{H})$'s on the right-hand sides of (3.15)–(3.17) are mode fields given by (A.3), (A.14), and (A.15). Substituting (A.3), (A.14), and (A.15) into (3.15)–(3.17), we obtain

$$E^{(1)}(0, \hat{M}_{pq}^{1TM}) = C_{pq}^{TM-} \left[-Z_{pq}^{TM} \underline{\epsilon}_{pq}^{TM}(y^+, z^+) + \underline{u}_x \frac{k_{pq}^2 \psi_{pq}^{TM}(y^+, z^+)}{j\omega\epsilon} \right] e^{\gamma_{pq}x} \quad (3.18)$$

$$H^{(1)}(0, \hat{M}_{pq}^{1TM}) = C_{pq}^{TM-} \underline{h}_{pq}^{TM}(y^+, z^+) e^{\gamma_{pq}x} \quad (3.19)$$

$$E^{(1)}(0, \hat{M}_{10}^{1TE}) = (C_{10}^{TE+} e^{-\gamma_{10}x} + C_{10}^{TE-} e^{\gamma_{10}x}) \underline{\epsilon}_{10}^{TE}(y^+, z^+) \quad (3.20)$$

$$H^{(1)}(0, \hat{M}_{10}^{1TE}) = (C_{10}^{TE+} e^{-\gamma_{10}x} - C_{10}^{TE-} e^{\gamma_{10}x}) Y_{10}^{TE} \underline{h}_{10}^{TE}(y^+, z^+) + \underline{u}_x (C_{10}^{TE+} e^{-\gamma_{10}x} + C_{10}^{TE-} e^{\gamma_{10}x}) \frac{k_{10}^2 \psi_{10}^{TE}(y^+, z^+)}{j\omega\mu} \quad (3.21)$$

$$E^{(1)}(0, \hat{M}^{1TE}) = C_{pq}^{TE-} \underline{\epsilon}_{pq}^{TE}(y^+, z^+) e^{\gamma_{pq}x}, \quad (p, q) \neq (1, 0) \quad (3.22)$$

$$H^{(1)}(0, \hat{M}_{pq}^{1TE}) = C_{pq}^{TE-} \left[-Y_{pq}^{TE} \underline{h}_{pq}^{TE}(y^+, z^+) + \underline{u}_x \frac{k_{pq}^2 \psi_{pq}^{TE}(y^+, z^+)}{j\omega\mu} \right] e^{\gamma_{pq}x} \quad (3.23)$$

Setting, as required by (3.14), the transverse part of $E^{(1)}(0, \hat{M}_{pq}^{1\delta})$ equal to $\underline{\epsilon}_{pq}^{\delta}(y^+, z^+)$ when $x = -x_0$, we obtain

$$C_{pq}^{TM-} = -\frac{e^{\gamma_{pq}x_0}}{Z_{pq}^{TM}} \quad (3.24)$$

$$1 = C_{10}^{TE+} e^{\gamma_{10}x_0} + C_{10}^{TE-} e^{-\gamma_{10}x_0} \quad (3.25)$$

$$C_{pq}^{TE-} = e^{\gamma_{pq}x_0}, \quad (p, q) \neq (1, 0) \quad (3.26)$$

The presence of Z_1 requires that, at $x = -L_1$, the ratio of the coefficient of $\underline{\epsilon}_{10}^{TE}(y^+, z^+)$ in (3.20) to the coefficient of $\underline{h}_{10}^{TE}(y^+, z^+)$ in (3.21) be $-Z_1$:

$$-Z_1 Y_{10}^{TE} = \frac{C_{10}^{TE+} e^{\gamma_{10}L_1} + C_{10}^{TE-} e^{-\gamma_{10}L_1}}{C_{10}^{TE+} e^{\gamma_{10}L_1} - C_{10}^{TE-} e^{-\gamma_{10}L_1}} \quad (3.27)$$

The constants C_{10}^{TE+} and C_{10}^{TE-} that satisfy (3.25) and (3.27) are

$$C_{10}^{TE+} = \frac{(Z_1 Y_{10}^{TE} - 1) e^{-\gamma_{10}L_1}}{2 \{ \sinh(\gamma_{10}(L_1 - x_0)) + Z_1 Y_{10}^{TE} \cosh(\gamma_{10}(L_1 - x_0)) \}} \quad (3.28)$$

$$C_{10}^{TE-} = \frac{(Z_1 Y_{10}^{TE} + 1) e^{\gamma_{10}L_1}}{2 \{ \sinh(\gamma_{10}(L_1 - x_0)) + Z_1 Y_{10}^{TE} \cosh(\gamma_{10}(L_1 - x_0)) \}} \quad (3.29)$$

In view of (A.13) and (A.25), substitution of (3.24), (3.26), (3.28), and (3.29) into (3.18)–(3.23) gives

$$E^{(1)}(Q, \hat{M}_{pq}^{1TM}) = \left(\underline{e}_{pq}^{TM}(y^+, z^+) - \underline{u}_x \frac{k_{pq}^2 \psi_{pq}^{TM}(y^+, z^+)}{\gamma_{pq}} \right) e^{\gamma_{pq}(x + x_o)} \quad (3.30)$$

$$H^{(1)}(Q, \hat{M}_{pq}^{1TM}) = -\frac{j\omega\epsilon}{\gamma_{pq}} \underline{h}_{pq}^{TM}(y^+, z^+) e^{\gamma_{pq}(x + x_o)} \quad (3.31)$$

$$E^{(1)}(Q, \hat{M}_{10}^{1TE}) = \underline{e}_{10}^{TE}(y^+, z^+) \frac{\sinh(\gamma_{10}(L_1 + x)) + Z_1 Y_{10}^{TE} \cosh(\gamma_{10}(L_1 + x))}{\sinh(\gamma_{10}(L_1 - x_o)) + Z_1 Y_{10}^{TE} \cosh(\gamma_{10}(L_1 - x_o))} \quad (3.32)$$

$$H^{(1)}(Q, \hat{M}_{10}^{1TE}) = \underline{h}_{10}^{TE}(y^+, z^+) \frac{j\gamma_{10}(\cosh(\gamma_{10}(L_1 + x)) + Z_1 Y_{10}^{TE} \sinh(\gamma_{10}(L_1 + x)))}{\omega\mu(\sinh(\gamma_{10}(L_1 - x_o)) + Z_1 Y_{10}^{TE} \cosh(\gamma_{10}(L_1 - x_o)))} - \underline{u}_x \cdot \psi_{10}^{TE}(y^+, z^+) \frac{j k_{10}^2 (\sinh(\gamma_{10}(L_1 + x)) + Z_1 Y_{10}^{TE} \cosh(\gamma_{10}(L_1 + x)))}{\omega\mu(\sinh(\gamma_{10}(L_1 - x_o)) + Z_1 Y_{10}^{TE} \cosh(\gamma_{10}(L_1 - x_o)))} \quad (3.33)$$

$$E^{(1)}(Q, \hat{M}_{pq}^{1TE}) = \underline{e}_{pq}^{TE}(y^+, z^+) e^{\gamma_{pq}(x + x_o)}, (p, q) \neq (1, 0) \quad (3.34)$$

$$H^{(1)}(Q, \hat{M}_{pq}^{1TE}) = \frac{j\gamma_{pq}}{\omega\mu} \left(\underline{h}_{pq}^{TE}(y^+, z^+) - \underline{u}_x \frac{k_{pq}^2 \psi_{pq}^{TE}(y^+, z^+)}{\gamma_{pq}} \right) e^{\gamma_{pq}(x + x_o)}, (p, q) \neq (1, 0) \quad (3.35)$$

The electromagnetic field in (3.30) and (3.31) is a $-x$ traveling wave (A.3), that in (3.32) and (3.33) is a combination of x traveling and $-x$ traveling waves (A.14) and (A.15), and that in (3.34) and (3.35) is a $-x$ traveling wave (A.15). When $x = -x_o$, the transverse part of the electric field (3.30) is \underline{e}_{pq}^{TM} , the electric field (3.32) is \underline{e}_{10}^{TE} , and the transverse part of the electric field (3.34) is \underline{e}_{pq}^{TE} . When $x = -L_1$, the ratio of the TE_{10} voltage associated with (3.32) to the TE_{10} current associated with (3.33) is $-Z_1$.

In a development similar to that in the previous three paragraphs, we obtain

$$E^{(2)}(Q, \hat{M}_{pq}^{2TM}) = \left(\underline{e}_{pq}^{TM}(z^+, y^+) + \underline{u}_x \frac{k_{pq}^2 \psi_{pq}^{TM}(y^+, z^+)}{\gamma_{pq}} \right) e^{-\gamma_{pq}(x - x_o)} \quad (3.36)$$

$$H^{(2)}(Q, \hat{M}_{pq}^{2TM}) = \frac{j\omega\epsilon}{\gamma_{pq}} h_{pq}^{TM}(y^+, z^+) e^{-\gamma_{pq}(x-x_o)} \quad (3.37)$$

$$E^{(2)}(Q, \hat{M}_{10}^{2TE}) = \underline{e}_{10}^{TE}(y^+, z^+) \frac{\sinh(\gamma_{10}(L_2 - x)) + Z_2 Y_{10}^{TE} \cosh(\gamma_{10}(L_2 - x))}{\sinh(\gamma_{10}(L_2 - x_o)) + Z_2 Y_{10}^{TE} \cosh(\gamma_{10}(L_2 - x_o))} \quad (3.38)$$

$$H^{(2)}(Q, \hat{M}_{10}^{2TE}) = -\underline{h}_{10}^{TE}(y^+, z^+) \frac{j\gamma_{10}(\cosh(\gamma_{10}(L_2 - x)) + Z_2 Y_{10}^{TE} \sinh(\gamma_{10}(L_2 - x)))}{\omega\mu(\sinh(\gamma_{10}(L_2 - x_o)) + Z_2 Y_{10}^{TE} \cosh(\gamma_{10}(L_2 - x_o)))} - \underline{u}_x \cdot \psi_{10}^{TE}(y^+, z^+) \frac{j k_{10}^2 (\sinh(\gamma_{10}(L_2 - x)) + Z_2 Y_{10}^{TE} \cosh(\gamma_{10}(L_2 - x)))}{\omega\mu(\sinh(\gamma_{10}(L_2 - x_o)) + Z_2 Y_{10}^{TE} \cosh(\gamma_{10}(L_2 - x_o)))} \quad (3.39)$$

$$E^{(2)}(Q, \hat{M}_{pq}^{2TE}) = \underline{e}_{pq}^{TE}(y^+, z^+) e^{-\gamma_{pq}(x-x_o)}, (p, q) \neq (1, 0) \quad (3.40)$$

$$H^{(2)}(Q, \hat{M}_{pq}^{2TE}) = -\frac{j\gamma_{pq}}{\omega\mu} \left(\underline{h}_{pq}^{TE}(y^+, z^+) + \underline{u}_x \frac{k_{pq}^2 \psi_{pq}^{TE}(y^+, z^+)}{\gamma_{pq}} \right) e^{-\gamma_{pq}(x-x_o)}, (p, q) \neq (1, 0) \quad (3.41)$$

The electromagnetic field in (3.36) and (3.37) is an x traveling wave (A.2), that in (3.38) and (3.39) is a combination of x traveling and $-x$ traveling waves (A.14) and (A.15), and that in (3.40) and (3.41) is an x traveling wave (A.14). When $x = x_o$, the transverse part of the electric field (3.36) is \underline{e}_{pq}^{TM} , the electric field (3.38) is \underline{e}_{10}^{TE} , and the electric field (3.40) is \underline{e}_{pq}^{TE} . When $x = L_2$, the ratio of the TE_{10} voltage associated with (3.38) to the TE_{10} current associated with (3.39) is Z_2 .

Suitable expressions for the electric and magnetic fields due to the \hat{M} 's of (3.1) and (3.2) are given in the previous two paragraphs. We are nearly ready to substitute these \hat{M} 's and their magnetic fields into (3.1) and (3.2). Letting $\delta = TM$ in (3.3) and using (A.4) and (A.5), we obtain

$$\hat{M}_{mn}^{1TM} = \underline{h}_{mn}^{TM}(y^+, z^+) \quad (3.42)$$

Letting $\delta = TE$ in (3.3) and using (A.16) and (A.17), we obtain

$$\hat{M}_{mn}^{1TE} = \underline{h}_{mn}^{TE}(y^+, z^+) \quad (3.43)$$

Substitution of (3.42), (3.43), (3.31), (3.33), and (3.35), into (3.1) and subsequent application of the orthogonality (A.26) give

$$\hat{Y}_{ij}^{1,1TM,1TM} = \frac{j\omega\epsilon\delta_{ij}}{\gamma_{pq}} \quad (3.44)$$

$$\hat{Y}_{ij}^{1,1TE,1TM} = 0 \quad (3.45)$$

$$\hat{Y}_{ij}^{1,1TM,1TE} = 0 \quad (3.46)$$

$$\begin{aligned} & \hat{Y}_{ij}^{1,1TE,1TE} \\ &= \begin{cases} -\frac{j\gamma_{10}(\cosh(\gamma_{10}(L_1 - x_o)) + Z_1 Y_{10}^{TE} \sinh(\gamma_{10}(L_1 - x_o)))\delta_{ij}}{\omega\mu(\sinh(\gamma_{10}(L_1 - x_o)) + Z_1 Y_{10}^{TE} \cosh(\gamma_{10}(L_1 - x_o)))}, & (p, q) = (1, 0) \\ -\frac{j\gamma_{pq}\delta_{ij}}{\omega\mu}, & (p, q) \neq (1, 0) \end{cases} \end{aligned} \quad (3.47)$$

where δ_{ij} is the Kronecker delta function:

$$\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases} \quad (3.48)$$

In (3.44) and (3.47), the subscript j is related to pq as in (2.26). The subscript j is not to be confused with the other j in (3.44) and (3.47). This other j is $\sqrt{-1}$.

In a development similar to that in the previous paragraph, we obtain

$$\hat{Y}_{ij}^{2,2TM,2TM} = \frac{j\omega\epsilon\delta_{ij}}{\gamma_{pq}} \quad (3.49)$$

$$\hat{Y}_{ij}^{2,2TE,2TM} = 0 \quad (3.50)$$

$$\hat{Y}_{ij}^{2,2TM,2TE} = 0 \quad (3.51)$$

$$\begin{aligned} & \hat{Y}_{ij}^{2,2TE,2TE} \\ &= \begin{cases} -\frac{j\gamma_{10}(\cosh(\gamma_{10}(L_2 - x_o)) + Z_2 Y_{10}^{TE} \sinh(\gamma_{10}(L_2 - x_o)))\delta_{ij}}{\omega\mu(\sinh(\gamma_{10}(L_2 - x_o)) + Z_2 Y_{10}^{TE} \cosh(\gamma_{10}(L_2 - x_o)))}, & (p, q) = (1, 0) \\ -\frac{j\gamma_{pq}\delta_{ij}}{\omega\mu}, & (p, q) \neq (1, 0) \end{cases} \end{aligned} \quad (3.52)$$

Chapter 4

The Admittance Matrix for the Circular Waveguide

In this chapter, $Y_{ij}^{3,\alpha\beta,\gamma\delta}$ of (2.30) is evaluated. The field $(\underline{E}^{(3)}(\underline{Q}, \underline{M}_{pq}^{\gamma\delta}), \underline{H}^{(3)}(\underline{Q}, \underline{M}_{pq}^{\gamma\delta}))$ is radiated by the magnetic current $\underline{M}_{pq}^{\gamma\delta}$ in the circular waveguide with the apertures A_1 and A_2 closed by perfect conductors, with the short at $z = L_3$, and with the matched load at the other end. Otherwise stated, $(\underline{E}^{(3)}(\underline{Q}, \underline{M}_{pq}^{\gamma\delta}), \underline{H}^{(3)}(\underline{Q}, \underline{M}_{pq}^{\gamma\delta}))$ is the field that would exist in region 3 of Fig. 2.1 if $\underline{J}^{\text{imp}}$, $-\underline{M}^{(1)}$, and $-\underline{M}^{(2)}$ were removed and if $\underline{M}_{pq}^{\gamma\delta}$ were put where $-\underline{M}^{(\gamma)}$ was. Our first objective is to obtain expressions for $\underline{E}^{(3)}(\underline{Q}, \underline{M}_{pq}^{\gamma\delta})$ and $\underline{H}^{(3)}(\underline{Q}, \underline{M}_{pq}^{\gamma\delta})$. An expression for $\underline{E}^{(3)}(\underline{Q}, \underline{M}_{pq}^{\gamma\delta})$ is needed to evaluate, as indicated in the last two sentences of Chapter 2, the field in the circular waveguide.

The volume density of magnetic current corresponding to the surface density $\underline{M}_{pq}^{\gamma\delta}(\phi, z)$ is $\underline{M}_{pq}^{\gamma\delta}(\phi, z)\delta(\rho - a)$ where $\delta(\rho - a)$ is the one-dimensional Dirac delta function. This volume density is expressed as

$$\underline{M}_{pq}^{\gamma\delta}(\phi, z)\delta(\rho - a) = \underline{u}_\phi M_{\phi pq}^{\gamma\delta}(\rho, \phi, z) + \underline{u}_z M_{zpq}^{\gamma\delta}(\rho, \phi, z) \quad (4.1)$$

where

$$M_{\phi pq}^{\gamma\delta}(\rho, \phi, z) = (\underline{u}_\phi \cdot \underline{M}_{pq}^{\gamma\delta}(\phi, z))\delta(\rho - a) \quad (4.2)$$

$$M_{zpq}^{\gamma\delta}(\rho, \phi, z) = (\underline{u}_z \cdot \underline{M}_{pq}^{\gamma\delta}(\phi, z))\delta(\rho - a) \quad (4.3)$$

Applying the Green's function technique, we obtain

$$\begin{aligned} \underline{E}^{(3)}(\underline{0}, \underline{M}_{pq}^{\gamma\delta}) &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dz' \int_0^{2\pi} d\phi' \int_0^a \rho' d\rho' \{ M_{\phi pq}^{\gamma\delta}(\rho', \phi', z') \\ &\cdot \hat{\underline{E}}^{(3)}(\underline{0}, \underline{u}_\phi \delta(\underline{r} - \underline{r}')) + M_{zpq}^{\gamma\delta}(\rho', \phi', z') \hat{\underline{E}}^{(3)}(\underline{0}, \underline{u}_z \delta(\underline{r} - \underline{r}')) \} \end{aligned} \quad (4.4)$$

$$\begin{aligned} \underline{H}^{(3)}(\underline{0}, \underline{M}_{pq}^{\gamma\delta}) &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dz' \int_0^{2\pi} d\phi' \int_0^a \rho' d\rho' \{ M_{\phi pq}^{\gamma\delta}(\rho', \phi', z') \\ &\cdot \hat{\underline{H}}^{(3)}(\underline{0}, \underline{u}_\phi \delta(\underline{r} - \underline{r}')) + M_{zpq}^{\gamma\delta}(\rho', \phi', z') \hat{\underline{H}}^{(3)}(\underline{0}, \underline{u}_z \delta(\underline{r} - \underline{r}')) \} \end{aligned} \quad (4.5)$$

where the operator $\hat{\underline{E}}^{(3)}$ differs from the operator $\underline{E}^{(3)}$ only in that the second argument of $\hat{\underline{E}}^{(3)}$ is a volume density instead of a surface density. Similarly, the superscript " \wedge " in $\hat{\underline{H}}^{(3)}$ indicates that the second argument is a volume density. In (4.4) and (4.5), $\delta(\underline{r} - \underline{r}')$ is the three-dimensional Dirac delta function, \underline{r} is the radius vector to the point (ρ, ϕ, z) at which $\underline{E}^{(3)}(\underline{0}, \underline{M}_{pq}^{\gamma\delta})$ and $\underline{H}^{(3)}(\underline{0}, \underline{M}_{pq}^{\gamma\delta})$ are evaluated, and \underline{r}' is the radius vector to the point (ρ', ϕ', z') .

The fields $\hat{\underline{E}}^{(3)}(\underline{0}, \underline{u}_\phi \delta(\underline{r} - \underline{r}'))$ and $\hat{\underline{H}}^{(3)}(\underline{0}, \underline{u}_\phi \delta(\underline{r} - \underline{r}'))$ are obtained by adding to the electromagnetic field of (D.19) and (D.20) the reflection due to the short at $z = L_3$. The electromagnetic field of (D.19) and (D.20) consists of a sum of fields each of which is of the form $(\underline{E}, \underline{H})$ where

$$\underline{E} = (\epsilon(z, z') \underline{E}_{\text{tan}} + \underline{u}_z E_z) e^{-\gamma|z-z'|} \quad (4.6)$$

$$\underline{H} = (\underline{H}_{\text{tan}} + \underline{u}_z \epsilon(z, z') H_z) e^{-\gamma|z-z'|} \quad (4.7)$$

where $\underline{E}_{\text{tan}}$ and $\underline{H}_{\text{tan}}$ have only ρ and ϕ components. Furthermore, $\epsilon(z, z')$ is given by (D.4), and γ is either γ_{np}^{TM} of (B.24) or γ_{np}^{TE} of (B.53). In this paragraph and in the next three paragraphs, the index p that appears in (D.19), (D.20), γ_{np}^{TM} , and γ_{np}^{TE} is not to be confused with the index p that appears in $\underline{M}_{pq}^{\gamma\delta}$, $M_{\phi pq}^{\gamma\delta}$ and $M_{zpq}^{\gamma\delta}$. Similarly, the index n that appears in (D.19), (D.20), γ_{np}^{TM} and γ_{np}^{TE} is not to be confused with the index n that appears in $\underline{M}_{mn}^{\alpha\beta}$. The reflected field due to $(\underline{E}, \underline{H})$ is a field $(\underline{E}^r, \underline{H}^r)$ which is proportional to $(\underline{E}, \underline{H})_{z < z'}$. Here, the subscript $z < z'$ denotes evaluation at $z < z'$. Adjusting the amplitude of $(\underline{E}^r, \underline{H}^r)$ so that the tangential component of $\underline{E} + \underline{E}^r$ vanishes when $z = L_3$, we obtain

$$\underline{E}^r = (-\underline{E}_{\text{tan}} + \underline{u}_z E_z) e^{\gamma(z+z'-2L_3)} \quad (4.8)$$

$$\underline{H}^r = (\underline{H}_{\text{tan}} - \underline{u}_z H_z) e^{\gamma(z+z'-2L_3)} \quad (4.9)$$

The sum of (4.6) and (4.8) is

$$\underline{E} + \underline{E}^r = 2e^{-\gamma(L_3-z')} \{ \underline{E}_{\text{tan}} \sinh(\gamma(L_3-z)) + \underline{u}_z E_z \cosh(\gamma(L_3-z)) \}, \quad z > z' \quad (4.10)$$

$$\underline{E} + \underline{E}^r = 2e^{-\gamma(L_3-z)} \cosh(\gamma(L_3-z')) \{ -\underline{E}_{\text{tan}} + \underline{u}_z E_z \}, \quad z < z' \quad (4.11)$$

The sum of (4.7) and (4.9) is

$$\underline{H} + \underline{H}^r = 2e^{-\gamma(L_3-z')} \{ \underline{H}_{\text{tan}} \cosh(\gamma(L_3-z)) + \underline{u}_z H_z \sinh(\gamma(L_3-z)) \}, \quad z > z' \quad (4.12)$$

$$\underline{H} + \underline{H}^r = 2e^{-\gamma(L_3-z)} \cosh(\gamma(L_3-z')) \{ \underline{H}_{\text{tan}} - \underline{u}_z H_z \}, \quad z < z' \quad (4.13)$$

In this paragraph, we have shown that the short at $z = L_3$ changes the field $(\underline{E}, \underline{H})$ that would exist in the circular waveguide matched at both ends to the field $(\underline{E} + \underline{E}^r, \underline{H} + \underline{H}^r)$.

Replacing each term of the form (4.6) in (D.19) by a term of the form (4.10) or (4.11), we obtain

$$\begin{aligned} & \hat{\underline{E}}^{(3)}(\underline{Q}, \underline{u}_\phi \delta(\underline{r} - \underline{r}')) \\ &= \frac{1}{\pi} \sum_{n=0}^{\infty} \sum_{p=1}^{\infty} \frac{\epsilon_n (k_{np}^{TM})^2 J'_n(k_{np}^{TM} \rho') e^{-\gamma_{np}^{TM}(L_3-z')}}{x_{np}^2 J_{n+1}^2(x_{np})} \\ & \cdot \left\{ -\underline{u}_\rho J'_n(k_{np}^{TM} \rho) \cos(n(\phi - \phi')) \sinh(\gamma_{np}^{TM}(L_3-z)) \right. \\ & + \underline{u}_\phi \frac{n J_n(k_{np}^{TM} \rho) \sin(n(\phi - \phi')) \sinh(\gamma_{np}^{TM}(L_3-z))}{k_{np}^{TM} \rho} \\ & + \underline{u}_z \frac{k_{np}^{TM} J_n(k_{np}^{TM} \rho) \cos(n(\phi - \phi')) \cosh(\gamma_{np}^{TM}(L_3-z))}{\gamma_{np}^{TM}} \left. \right\} \\ & + \frac{2}{\pi} \sum_{n=1}^{\infty} \sum_{p=1}^{\infty} \frac{(k_{np}^{TE})^2 n J_n(k_{np}^{TE} \rho') e^{-\gamma_{np}^{TE}(L_3-z')} \sinh(\gamma_{np}^{TE}(L_3-z))}{(x_{np}'^2 - n^2) (k_{np}^{TE} \rho') J_n^2(x_{np})} \\ & \cdot \left\{ -\underline{u}_\rho \frac{n J_n(k_{np}^{TE} \rho) \cos(n(\phi - \phi'))}{k_{np}^{TE} \rho} + \underline{u}_\phi J'_n(k_{np}^{TE} \rho) \sin(n(\phi - \phi')) \right\}, \\ & \qquad \qquad \qquad z > z' \quad (4.14) \end{aligned}$$

$$\begin{aligned}
& \hat{E}^{(3)}(\mathbf{Q}, \underline{u}_\phi \delta(\underline{r} - \underline{r}')) \\
&= \frac{1}{\pi} \sum_{n=0}^{\infty} \sum_{p=1}^{\infty} \frac{\epsilon_n (k_{np}^{TM})^2 J'_n(k_{np}^{TM} \rho') e^{-\gamma_{np}^{TM}(L_3 - z)} \cosh(\gamma_{np}^{TM}(L_3 - z'))}{x_{np}^2 J_{n+1}^2(x_{np})} \\
& \cdot \left\{ \underline{u}_\rho J'_n(k_{np}^{TM} \rho) \cos(n(\phi - \phi')) - \underline{u}_\phi \frac{n J_n(k_{np}^{TM} \rho) \sin(n(\phi - \phi'))}{k_{np}^{TM} \rho} \right. \\
& \left. + \underline{u}_z \frac{k_{np}^{TM} J_n(k_{np}^{TM} \rho) \cos(n(\phi - \phi'))}{\gamma_{np}^{TM}} \right\} \\
& + \frac{2}{\pi} \sum_{n=1}^{\infty} \sum_{p=1}^{\infty} \frac{(\gamma_{np}^{TE})^2 n J_n(k_{np}^{TE} \rho') e^{-\gamma_{np}^{TE}(L_3 - z)} \cosh(\gamma_{np}^{TE}(L_3 - z'))}{(x_{np}'^2 - n^2) (k_{np}^{TE} \rho') J_n^2(x_{np}')} \\
& \cdot \left\{ \underline{u}_\rho \frac{n J_n(k_{np}^{TE} \rho) \cos(n(\phi - \phi'))}{k_{np}^{TE} \rho} - \underline{u}_\phi J'_n(k_{np}^{TE} \rho) \sin(n(\phi - \phi')) \right\}, \\
& \qquad \qquad \qquad z < z' \qquad (4.15)
\end{aligned}$$

Replacing each term of the form (4.7) in (D.20) by a term of the form (4.12) or (4.13), we obtain

$$\begin{aligned}
& \hat{H}^{(3)}(\mathbf{Q}, \underline{u}_\phi \delta(\underline{r} - \underline{r}')) \\
&= \frac{-jk^2}{\pi \omega \mu} \sum_{n=0}^{\infty} \sum_{p=1}^{\infty} \frac{\epsilon_n (k_{np}^{TM})^2 J'_n(k_{np}^{TM} \rho') e^{-\gamma_{np}^{TM}(L_3 - z')} \cosh(\gamma_{np}^{TM}(L_3 - z))}{\gamma_{np}^{TM} x_{np}^2 J_{n+1}^2(x_{np})} \\
& \cdot \left\{ \underline{u}_\rho \frac{n J_n(k_{np}^{TM} \rho) \sin(n(\phi - \phi'))}{k_{np}^{TM} \rho} + \underline{u}_\phi J'_n(k_{np}^{TM} \rho) \cos(n(\phi - \phi')) \right\} \\
& + \frac{j2}{\pi \omega \mu} \sum_{n=1}^{\infty} \sum_{p=1}^{\infty} \frac{\gamma_{np}^{TE} (k_{np}^{TE})^2 n J_n(k_{np}^{TE} \rho') e^{-\gamma_{np}^{TE}(L_3 - z')} \cosh(\gamma_{np}^{TE}(L_3 - z))}{(x_{np}'^2 - n^2) (k_{np}^{TE} \rho') J_n^2(x_{np}')} \\
& \cdot \left\{ (\underline{u}_\rho J'_n(k_{np}^{TE} \rho) \sin(n(\phi - \phi')) + \underline{u}_\phi \frac{n J_n(k_{np}^{TE} \rho) \cos(n(\phi - \phi'))}{k_{np}^{TE} \rho}) \right. \\
& \cdot \cosh(\gamma_{np}^{TE}(L_3 - z)) \\
& \left. - \underline{u}_z \frac{k_{np}^{TE} J_n(k_{np}^{TE} \rho) \sin(n(\phi - \phi')) \sinh(\gamma_{np}^{TE}(L_3 - z))}{\gamma_{np}^{TE}} \right\}, z > z' \quad (4.16)
\end{aligned}$$

$$\begin{aligned}
& \hat{H}^{(3)}(\mathbf{Q}, \underline{u}_\phi \delta(\mathbf{r} - \mathbf{r}')) \\
&= \frac{-jk^2}{\pi\omega\mu} \sum_{n=0}^{\infty} \sum_{p=1}^{\infty} \frac{\epsilon_n(k_{np}^{TM})^2 J'_n(k_{np}^{TM} \rho') e^{-\gamma_{np}^{TM}(L_3-z)} \cosh(\gamma_{np}^{TM}(L_3-z'))}{\gamma_{np}^{TM} x_{np}^2 J_{n+1}^2(x_{np})} \\
&\quad \cdot \left\{ \underline{u}_\rho \frac{n J_n(k_{np}^{TM} \rho) \sin(n(\phi - \phi'))}{k_{np}^{TM} \rho} + \underline{u}_\phi J'_n(k_{np}^{TM} \rho) \cos(n(\phi - \phi')) \right\} \\
&\quad + \frac{j2}{\pi\omega\mu} \sum_{n=1}^{\infty} \sum_{p=1}^{\infty} \frac{\gamma_{np}^{TE} (k_{np}^{TE})^2 n J_n(k_{np}^{TE} \rho') e^{-\gamma_{np}^{TE}(L_3-z)} \cosh(\gamma_{np}^{TE}(L_3-z'))}{(x_{np}'^2 - n^2) (k_{np}^{TE} \rho') J_n^2(x_{np}') } \\
&\quad \cdot \left\{ \underline{u}_\rho J'_n(k_{np}^{TE} \rho) \sin(n(\phi - \phi')) \right. \\
&\quad \left. + \underline{u}_\phi \frac{n J_n(k_{np}^{TE} \rho) \cos(n(\phi - \phi'))}{k_{np}^{TE} \rho} \right. \\
&\quad \left. + \underline{u}_z \frac{k_{np}^{TE} J_n(k_{np}^{TE} \rho) \sin(n(\phi - \phi'))}{\gamma_{np}^{TE}} \right\}, z < z' \tag{4.17}
\end{aligned}$$

The fields $\hat{E}^{(3)}(\mathbf{Q}, \underline{u}_z \delta(\mathbf{r} - \mathbf{r}'))$ and $\hat{H}^{(3)}(\mathbf{Q}, \underline{u}_z \delta(\mathbf{r} - \mathbf{r}'))$ are obtained by adding to the electromagnetic field of (D.55) and (D.58) the reflection due to the short at $z = L_3$. Aside from the $\delta(z - z')$ term in (D.58), the electromagnetic field of (D.55) and (D.58) consists of a sum of fields, each of which is of the form $(\underline{E}^{TE}, \underline{H}^{TE})$ where

$$\underline{E}^{TE} = \underline{E}_{\tan}^{TE} e^{-\gamma_{np}^{TE}|z-z'|} \tag{4.18}$$

$$\underline{H}^{TE} = (\epsilon(z, z') \underline{H}_{\tan}^{TE} + \underline{u}_z H_z^{TE}) e^{-\gamma_{np}^{TE}|z-z'|} \tag{4.19}$$

where $\underline{E}_{\tan}^{TE}$ and $\underline{H}_{\tan}^{TE}$ have only ρ and ϕ components. The reflected field due to $(\underline{E}^{TE}, \underline{H}^{TE})$ is called $(\underline{E}^{TEr}, \underline{H}^{TEr})$ and is given by

$$\underline{E}^{TEr} = -\underline{E}_{\tan}^{TE} e^{\gamma_{np}^{TE}(z+z'-2L_3)} \tag{4.20}$$

$$\underline{H}^{TEr} = (\underline{H}_{\tan}^{TE} - \underline{u}_z H_z^{TE}) e^{\gamma_{np}^{TE}(z+z'-2L_3)} \tag{4.21}$$

The sum of (4.18) and (4.20) is

$$\underline{E}^{TE} + \underline{E}^{TEr} = 2\underline{E}_{\tan}^{TE} e^{-\gamma_{np}^{TE}(L_3-z')} \sinh(\gamma_{np}^{TE}(L_3-z)), z > z' \tag{4.22}$$

$$\underline{E}^{TE} + \underline{E}^{TEr} = 2\underline{E}_{\tan}^{TE} e^{-\gamma_{np}^{TE}(L_3-z)} \sinh(\gamma_{np}^{TE}(L_3-z')), z < z' \tag{4.23}$$

The sum of (4.19) and (4.21) is

$$\begin{aligned} \underline{H}^{TE} + \underline{H}^{TEr} &= 2e^{-\gamma_{np}^{TE}(L_3-z')} \{ \underline{H}_{\tan}^{TE} \cosh(\gamma_{np}^{TE}(L_3-z)) \\ &\quad + \underline{u}_z \underline{H}_z^{TE} \sinh(\gamma_{np}^{TE}(L_3-z)) \}, \quad z > z' \end{aligned} \quad (4.24)$$

$$\begin{aligned} \underline{H}^{TE} + \underline{H}^{TEr} &= 2e^{-\gamma_{np}^{TE}(L_3-z)} \sinh(\gamma_{np}^{TE}(L_3-z')) \\ &\quad \cdot \{ -\underline{H}_{\tan}^{TE} + \underline{u}_z \underline{H}_z^{TE} \}, \quad z < z' \end{aligned} \quad (4.25)$$

The short at $z = L_3$ changes the field $(\underline{E}^{TE}, \underline{H}^{TE})$ that would exist in the circular waveguide matched at both ends to the field $(\underline{E}^{TE} + \underline{E}^{TEr}, \underline{H}^{TE} + \underline{H}^{TEr})$.

Replacing each term of the form (4.18) in (D.55) by a term of the form (4.22) or (4.23), we obtain

$$\begin{aligned} &\hat{\underline{E}}^{(3)}(\underline{Q}, \underline{u}_z \delta(\underline{r} - \underline{r}')) \\ &= \frac{1}{\pi} \sum_{n=0}^{\infty} \sum_{p=1}^{\infty} \frac{\epsilon_n (k_{np}^{TE})^3 J_n(k_{np}^{TE} \rho') e^{-\gamma_{np}^{TE}(L_3-z')} \sinh(\gamma_{np}^{TE}(L_3-z))}{\gamma_{np}^{TE} (x_{np}'^2 - n^2) J_n^2(x_{np}') } \\ &\quad \cdot \{ \underline{u}_\rho \frac{n J_n(k_{np}^{TE} \rho) \sin(n(\phi - \phi'))}{k_{np}^{TE} \rho} + \underline{u}_\phi J_n'(k_{np}^{TE} \rho) \cos(n(\phi - \phi')) \}, \\ &\quad \quad \quad z > z' \end{aligned} \quad (4.26)$$

$$\begin{aligned} &\hat{\underline{E}}^{(3)}(\underline{Q}, \underline{u}_z \delta(\underline{r} - \underline{r}')) \\ &= \frac{1}{\pi} \sum_{n=0}^{\infty} \sum_{p=1}^{\infty} \frac{\epsilon_n (k_{np}^{TE})^3 J_n(k_{np}^{TE} \rho') e^{-\gamma_{np}^{TE}(L_3-z)} \sinh(\gamma_{np}^{TE}(L_3-z'))}{\gamma_{np}^{TE} (x_{np}'^2 - n^2) J_n^2(x_{np}') } \\ &\quad \cdot \{ \underline{u}_\rho \frac{n J_n(k_{np}^{TE} \rho) \sin(n(\phi - \phi'))}{k_{np}^{TE} \rho} + \underline{u}_\phi J_n'(k_{np}^{TE} \rho) \cos(n(\phi - \phi')) \}, \\ &\quad \quad \quad z < z' \end{aligned} \quad (4.27)$$

Replacing each term of the form (4.19) in (D.58) by a term of the form (4.24) or (4.25) and retaining the $\delta(z - z')$ term in (D.58), we obtain

$$\begin{aligned} \hat{\underline{H}}^{(3)}(\underline{Q}, \underline{u}_z \delta(\underline{r} - \underline{r}')) &= \frac{j}{\pi \omega \mu} \sum_{n=0}^{\infty} \sum_{p=1}^{\infty} \frac{\epsilon_n (k_{np}^{TE})^3 J_n(k_{np}^{TE} \rho')}{(x_{np}'^2 - n^2) J_n^2(x_{np}') } \\ &\quad \cdot \left[\underline{u}_z \delta(z - z') \frac{J_n(k_{np}^{TE} \rho) \cos(n(\phi - \phi'))}{k_{np}^{TE}} + \left\{ \begin{array}{l} \underline{H}^+, z' > z \\ \underline{H}^-, z' < z \end{array} \right\} \right] \end{aligned} \quad (4.28)$$

where

$$H^- = \left\{ \left(\underline{u}_\rho J'_n(k_{np}^{TE} \rho) \cos(n(\phi - \phi')) - \underline{u}_\phi \frac{n J_n(k_{np}^{TE} \rho) \sin(n(\phi - \phi'))}{k_{np}^{TE} \rho} \right) \cdot \cosh(\gamma_{np}^{TE} (L_3 - z)) - \underline{u}_z \frac{k_{np}^{TE} J_n(k_{np}^{TE} \rho) \cos(n(\phi - \phi')) \sinh(\gamma_{np}^{TE} (L_3 - z))}{\gamma_{np}^{TE}} \right\} e^{-\gamma_{np}^{TE} (L_3 - z')} \quad (4.29)$$

$$H^+ = \left\{ -\underline{u}_\rho J'_n(k_{np}^{TE} \rho) \cos(n(\phi - \phi')) + \underline{u}_\phi \frac{n J_n(k_{np}^{TE} \rho) \sin(n(\phi - \phi'))}{k_{np}^{TE} \rho} - \underline{u}_z \frac{k_{np}^{TE} J_n(k_{np}^{TE} \rho) \cos(n(\phi - \phi'))}{\gamma_{np}^{TE}} \right\} e^{-\gamma_{np}^{TE} (L_3 - z)} \sinh(\gamma_{np}^{TE} (L_3 - z')) \quad (4.30)$$

Before substituting (4.14)–(4.17) and (4.26)–(4.28) into (4.4) and (4.5), we obtain expressions for $M_{\phi pq}^{\gamma\delta}$ and $M_{z pq}^{\gamma\delta}$. Substitution of (A.10) into (2.13) and (2.14) gives

$$M_{pq}^{\gamma TM}(\phi, z) = -\frac{2\pi}{k_{pq} \sqrt{bc}} \left\{ \underline{u}_\phi \frac{q}{c} \sin\left(\frac{p\pi y^{\gamma+}}{b}\right) \cos\left(\frac{q\pi z^+}{c}\right) - \underline{u}_z (-1)^\gamma \left(\frac{p}{b}\right) \left(\frac{\sin \phi_o}{\phi_o}\right) \cos\left(\frac{p\pi y^{\gamma+}}{b}\right) \sin\left(\frac{q\pi z^+}{c}\right) \right\} \quad (4.31)$$

where γ is either 1 or 2. Moreover, $y^{\gamma+}$ and z^+ are given by (2.15)–(2.17). Finally, from (A.8), k_{pq} is given by

$$k_{pq} = \sqrt{\left(\frac{p\pi}{b}\right)^2 + \left(\frac{q\pi}{c}\right)^2} \quad (4.32)$$

Substitution of (A.23) into (2.13) and (2.14) yields

$$M_{pq}^{\gamma TE}(\phi, z) = -\frac{2\pi}{k_{pq}} \sqrt{\frac{\epsilon_p \epsilon_q}{4bc}} \left\{ \underline{u}_\phi \frac{p}{b} \sin\left(\frac{p\pi y^{\gamma+}}{b}\right) \cos\left(\frac{q\pi z^+}{c}\right) + \underline{u}_z (-1)^\gamma \left(\frac{q}{c}\right) \left(\frac{\sin \phi_o}{\phi_o}\right) \cos\left(\frac{p\pi y^{\gamma+}}{b}\right) \sin\left(\frac{q\pi z^+}{c}\right) \right\} \quad (4.33)$$

where ϵ_p is Neumann's number given by (A.21). Substituting (4.31) into (4.2) and (4.3), we obtain

$$M_{\phi pq}^{\gamma TM}(\rho, \phi, z) = -\frac{2\pi}{k_{pq} \sqrt{bc}} \left(\frac{q}{c}\right) \sin\left(\frac{p\pi y^{\gamma+}}{b}\right) \cos\left(\frac{q\pi z^+}{c}\right) \delta(\rho - a) \quad (4.34)$$

$$M_{zpq}^{\gamma TM}(\rho, \phi, z) = \frac{2\pi(-1)^\gamma}{k_{pq}\sqrt{bc}} \left(\frac{p}{b}\right) \left(\frac{\sin \phi_o}{\phi_o}\right) \cos\left(\frac{p\pi y^{\gamma+}}{b}\right) \sin\left(\frac{q\pi z^+}{c}\right) \delta(\rho - a) \quad (4.35)$$

Substituting (4.33) into (4.2) and (4.3), we obtain

$$M_{\phi pq}^{\gamma TE}(\rho, \phi, z) = -\frac{\pi}{k_{pq}} \sqrt{\frac{\epsilon_p \epsilon_q}{bc}} \left(\frac{p}{b}\right) \sin\left(\frac{p\pi y^{\gamma+}}{b}\right) \cos\left(\frac{q\pi z^+}{c}\right) \delta(\rho - a) \quad (4.36)$$

$$M_{zpq}^{\gamma TE}(\rho, \phi, z) = -\frac{\pi(-1)^\gamma}{k_{pq}} \sqrt{\frac{\epsilon_p \epsilon_q}{bc}} \left(\frac{q}{c}\right) \left(\frac{\sin \phi_o}{\phi_o}\right) \cos\left(\frac{p\pi y^{\gamma+}}{b}\right) \cdot \sin\left(\frac{q\pi z^+}{c}\right) \delta(\rho - a) \quad (4.37)$$

Substituting (4.34) and (4.35) into (4.4) and performing the integration with respect to ρ' , we obtain

$$\begin{aligned} \underline{E}^{(3)}(\underline{Q}, \underline{M}_{pq}^{\gamma TM}) &= -\frac{2\pi a}{k_{pq}\sqrt{bc}} \int_{-\frac{\xi}{2}}^{\frac{\xi}{2}} dz' \int_{\phi_1}^{\phi_2} d\phi' \\ &\cdot \left\{ \left(\frac{q}{c}\right) \sin\left(\frac{p\pi y^{\gamma+}}{b}\right) \cos\left(\frac{q\pi z'^+}{c}\right) \left[\hat{\underline{E}}^{(3)}(\underline{Q}, \underline{u}_\phi \delta(\underline{r} - \underline{r}')) \right]_{\rho'=a} \right. \\ &\left. - (-1)^\gamma \left(\frac{p}{b}\right) \left(\frac{\sin \phi_o}{\phi_o}\right) \cos\left(\frac{p\pi y^{\gamma+}}{b}\right) \sin\left(\frac{q\pi z'^+}{c}\right) \left[\hat{\underline{E}}^{(3)}(\underline{Q}, \underline{u}_z \delta(\underline{r} - \underline{r}')) \right]_{\rho'=a} \right\} \quad (4.38) \end{aligned}$$

where γ is either 1 or 2 and, recalling (2.15)–(2.17),

$$y'^+ = (\pi - \phi')x_o + \frac{\xi}{2} \quad (4.39)$$

$$y'^+ = \phi'x_o + \frac{\xi}{2} \quad (4.40)$$

$$z'^+ = z' + \frac{\xi}{2} \quad (4.41)$$

$$\phi_1 = (2 - \gamma)\pi - \phi_o \quad (4.42)$$

$$\phi_2 = (2 - \gamma)\pi + \phi_o \quad (4.43)$$

Similar substitution of (4.36) and (4.37) into (4.4) gives

$$\begin{aligned} \underline{E}^{(3)}(\underline{Q}, \underline{M}_{pq}^{\gamma TE}) &= -\frac{\pi a}{k_{pq}} \sqrt{\frac{\epsilon_p \epsilon_q}{bc}} \int_{-\frac{\xi}{2}}^{\frac{\xi}{2}} dz' \int_{\phi_1}^{\phi_2} d\phi' \\ &\cdot \left\{ \left(\frac{p}{b}\right) \sin\left(\frac{p\pi y^{\gamma+}}{b}\right) \cos\left(\frac{q\pi z'^+}{c}\right) \left[\hat{\underline{E}}^{(3)}(\underline{Q}, \underline{u}_\phi \delta(\underline{r} - \underline{r}')) \right]_{\rho'=a} \right. \\ &\left. + (-1)^\gamma \left(\frac{q}{c}\right) \left(\frac{\sin \phi_o}{\phi_o}\right) \cos\left(\frac{p\pi y^{\gamma+}}{b}\right) \sin\left(\frac{q\pi z'^+}{c}\right) \left[\hat{\underline{E}}^{(3)}(\underline{Q}, \underline{u}_z \delta(\underline{r} - \underline{r}')) \right]_{\rho'=a} \right\} \quad (4.44) \end{aligned}$$

Since (4.5) is (4.4) with $\underline{E}^{(3)}$ and $\hat{\underline{E}}^{(3)}$ replaced by $\underline{H}^{(3)}$ and $\hat{\underline{H}}^{(3)}$, respectively, the equations obtained by replacing $\underline{E}^{(3)}$ and $\hat{\underline{E}}^{(3)}$ by $\underline{H}^{(3)}$ and $\hat{\underline{H}}^{(3)}$ in (4.38) and (4.44) are valid:

$$\begin{aligned} \underline{H}^{(3)}(\underline{Q}, \underline{M}_{pq}^{\gamma TM}) = & -\frac{2\pi a}{k_{pq}\sqrt{bc}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dz' \int_{\phi_1}^{\phi_2} d\phi' \\ & \cdot \left\{ \left(\frac{q}{c}\right) \sin\left(\frac{p\pi y'^{++}}{b}\right) \cos\left(\frac{q\pi z'^{++}}{c}\right) \left[\hat{\underline{H}}^{(3)}(\underline{Q}, \underline{u}_\phi \delta(\underline{r} - \underline{r}'))\right]_{\rho'=a} \right. \\ & \left. - (-1)^\gamma \left(\frac{p}{b}\right) \left(\frac{\sin \phi_o}{\phi_o}\right) \cos\left(\frac{p\pi y'^{++}}{b}\right) \sin\left(\frac{q\pi z'^{++}}{c}\right) \left[\hat{\underline{H}}^{(3)}(\underline{Q}, \underline{u}_z \delta(\underline{r} - \underline{r}'))\right]_{\rho'=a} \right\} \quad (4.45) \end{aligned}$$

$$\begin{aligned} \underline{H}^{(3)}(\underline{Q}, \underline{M}_{pq}^{\gamma TE}) = & -\frac{\pi a}{k_{pq}} \sqrt{\frac{\epsilon_p \epsilon_q}{bc}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dz' \int_{\phi_1}^{\phi_2} d\phi' \\ & \cdot \left\{ \left(\frac{p}{b}\right) \sin\left(\frac{p\pi y'^{++}}{b}\right) \cos\left(\frac{q\pi z'^{++}}{c}\right) \left[\hat{\underline{H}}^{(3)}(\underline{Q}, \underline{u}_\phi \delta(\underline{r} - \underline{r}'))\right]_{\rho'=a} \right. \\ & \left. + (-1)^\gamma \left(\frac{q}{c}\right) \left(\frac{\sin \phi_o}{\phi_o}\right) \cos\left(\frac{p\pi y'^{++}}{b}\right) \sin\left(\frac{q\pi z'^{++}}{c}\right) \left[\hat{\underline{H}}^{(3)}(\underline{Q}, \underline{u}_z \delta(\underline{r} - \underline{r}'))\right]_{\rho'=a} \right\} \quad (4.46) \end{aligned}$$

Replacing (n, p) in (4.14), (4.15), (4.26), and (4.27) by (r, s) and then substituting the resulting expressions into (4.38), we obtain

$$\begin{aligned} \underline{E}^{(3)}(\underline{Q}, \underline{M}_{pq}^{\gamma TM}) = & -\frac{4a}{k_{pq}\sqrt{bc}} \left\{ \frac{q}{c} \sum_{r=0}^{\infty} \sum_{s=1}^{\infty} \frac{\epsilon_r (k_{rs}^{TM})^2 J'_r(k_{rs}^{TM} a) \underline{E}^{\gamma TM\phi}}{2x_{rs}^2 J_{r+1}^2(x_{rs})} \right. \\ & + \frac{q}{c} \sum_{r=1}^{\infty} \sum_{s=1}^{\infty} \frac{(k_{rs}^{TE})^2 r J_r(k_{rs}^{TE} a) \underline{E}^{\gamma TE\phi}}{(k_{rs}^{TE} a)(x_{rs}'^2 - r^2) J_r^2(x_{rs}')} \\ & \left. - \left(\frac{p}{b}\right) \left(\frac{\sin \phi_o}{\phi_o}\right) \sum_{r=0}^{\infty} \sum_{s=1}^{\infty} \frac{\epsilon_r (k_{rs}^{TE})^3 J_r(k_{rs}^{TE} a) \underline{E}^{\gamma TEz}}{2\gamma_{rs}^{TE} (x_{rs}'^2 - r^2) J_r^2(x_{rs}')} \right\} \quad (4.47) \end{aligned}$$

where

$$\underline{E}^{\gamma TM\phi} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\left(\frac{q\pi z'^{++}}{c}\right) \left\{ \begin{array}{l} \underline{E}^{\gamma TM\phi+}, z' > z \\ \underline{E}^{\gamma TM\phi-}, z' < z \end{array} \right\} dz' \quad (4.48)$$

$$\underline{E}^{\gamma TE\phi} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\left(\frac{q\pi z'^{++}}{c}\right) \left\{ \begin{array}{l} \underline{E}^{\gamma TE\phi+}, z' > z \\ \underline{E}^{\gamma TE\phi-}, z' < z \end{array} \right\} dz' \quad (4.49)$$

$$\underline{E}^{\gamma TEz} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin\left(\frac{q\pi z'^{++}}{c}\right) \left\{ \begin{array}{l} \underline{E}^{\gamma TEz+}, z' > z \\ \underline{E}^{\gamma TEz-}, z' < z \end{array} \right\} dz' \quad (4.50)$$

In (4.48)–(4.50),

$$\begin{aligned} E^{\gamma TM\phi-} = & -e^{-\gamma_{rs}^{TM}(L_3-z')} \left\{ (\underline{u}_\rho \phi^{\gamma 2} J'_r(k_{rs}^{TM} \rho) - \underline{u}_\phi \frac{\phi^{\gamma 1} r J_r(k_{rs}^{TM} \rho)}{k_{rs}^{TM} \rho}) \right. \\ & \cdot \sinh(\gamma_{rs}^{TM}(L_3 - z)) - \underline{u}_z \frac{\phi^{\gamma 2} k_{rs}^{TM} J_r(k_{rs}^{TM} \rho) \cosh(\gamma_{rs}^{TM}(L_3 - z))}{\gamma_{rs}^{TM}} \left. \right\} \quad (4.51) \end{aligned}$$

$$\begin{aligned} E^{\gamma TM\phi+} = & \cosh(\gamma_{rs}^{TM}(L_3 - z')) e^{-\gamma_{rs}^{TM}(L_3-z)} \left\{ \underline{u}_\rho \phi^{\gamma 2} J'_r(k_{rs}^{TM} \rho) \right. \\ & - \underline{u}_\phi \frac{\phi^{\gamma 1} r J_r(k_{rs}^{TM} \rho)}{k_{rs}^{TM} \rho} + \underline{u}_z \frac{\phi^{\gamma 2} k_{rs}^{TM} J_r(k_{rs}^{TM} \rho)}{\gamma_{rs}^{TM}} \left. \right\} \quad (4.52) \end{aligned}$$

$$\begin{aligned} E^{\gamma TE\phi-} = & -e^{-\gamma_{rs}^{TE}(L_3-z')} \sinh(\gamma_{rs}^{TE}(L_3 - z)) \\ & \cdot \left\{ \underline{u}_\rho \frac{\phi^{\gamma 2} r J_r(k_{rs}^{TE} \rho)}{k_{rs}^{TE} \rho} - \underline{u}_\phi \phi^{\gamma 1} J'_r(k_{rs}^{TE} \rho) \right\} \quad (4.53) \end{aligned}$$

$$\begin{aligned} E^{\gamma TE\phi+} = & \cosh(\gamma_{rs}^{TE}(L_3 - z')) e^{-\gamma_{rs}^{TE}(L_3-z)} \\ & \cdot \left\{ \underline{u}_\rho \frac{\phi^{\gamma 2} r J_r(k_{rs}^{TE} \rho)}{k_{rs}^{TE} \rho} - \underline{u}_\phi \phi^{\gamma 1} J'_r(k_{rs}^{TE} \rho) \right\} \quad (4.54) \end{aligned}$$

$$\begin{aligned} E^{\gamma TEz-} = & e^{-\gamma_{rs}^{TE}(L_3-z')} \sinh(\gamma_{rs}^{TE}(L_3 - z)) (-1)^\gamma \\ & \cdot \left\{ \underline{u}_\rho \frac{\phi^{\gamma 3} r J_r(k_{rs}^{TE} \rho)}{k_{rs}^{TE} \rho} + \underline{u}_\phi \phi^{\gamma 4} J'_r(k_{rs}^{TE} \rho) \right\} \quad (4.55) \end{aligned}$$

$$\begin{aligned} E^{\gamma TEz+} = & \sinh(\gamma_{rs}^{TE}(L_3 - z')) e^{-\gamma_{rs}^{TE}(L_3-z)} (-1)^\gamma \\ & \cdot \left\{ \underline{u}_\rho \frac{\phi^{\gamma 3} r J_r(k_{rs}^{TE} \rho)}{k_{rs}^{TE} \rho} + \underline{u}_\phi \phi^{\gamma 4} J'_r(k_{rs}^{TE} \rho) \right\} \quad (4.56) \end{aligned}$$

where

$$\phi^{\gamma 1} = \int_{\phi_1}^{\phi_2} \sin\left(\frac{p\pi y^{\gamma+}}{b}\right) \sin(r(\phi - \phi')) d\phi' \quad (4.57)$$

$$\phi^{\gamma 2} = \int_{\phi_1}^{\phi_2} \sin\left(\frac{p\pi y^{\gamma+}}{b}\right) \cos(r(\phi - \phi')) d\phi' \quad (4.58)$$

$$\phi^{\gamma 3} = \int_{\phi_1}^{\phi_2} \cos\left(\frac{p\pi y^{\gamma+}}{b}\right) \sin(r(\phi - \phi')) d\phi' \quad (4.59)$$

$$\phi^{\gamma 4} = \int_{\phi_1}^{\phi_2} \cos\left(\frac{p\pi y^{\gamma+}}{b}\right) \cos(r(\phi - \phi')) d\phi' \quad (4.60)$$

Replacing (n, p) in (4.14), (4.15), (4.26), and (4.27) by (r, s) and then sub-

stituting the resulting expressions into (4.44), we obtain

$$E^{(3)}(Q, M_{pq}^{\gamma TE}) = -\frac{4a}{k_{pq}} \sqrt{\frac{\epsilon_p \epsilon_q}{4bc}} \left\{ \frac{p}{b} \sum_{r=0}^{\infty} \sum_{s=1}^{\infty} \frac{\epsilon_r (k_{rs}^{TM})^2 J'_r(k_{rs}^{TM} a) E^{\gamma TM \phi}}{2x_{rs}^2 J_{r+1}^2(x_{rs})} \right. \\ \left. + \frac{p}{b} \sum_{r=1}^{\infty} \sum_{s=1}^{\infty} \frac{(k_{rs}^{TE})^2 r J_r(k_{rs}^{TE} a) E^{\gamma TE \phi}}{(k_{rs}^{TE} a)(x_{rs}'^2 - r^2) J_r^2(x_{rs}') } \right. \\ \left. + \left(\frac{q}{c}\right) \left(\frac{\sin \phi_0}{\phi_0}\right) \sum_{r=0}^{\infty} \sum_{s=1}^{\infty} \frac{\epsilon_r (k_{rs}^{TE})^3 J_r(k_{rs}^{TE} a) E^{\gamma TE z}}{2\gamma_{rs}^{TE} (x_{rs}'^2 - r^2) J_r^2(x_{rs}') } \right\} \quad (4.61)$$

Replacement of (n, p) in (4.16), (4.17), and (4.28) by (r, s) and subsequent substitution of the resulting expressions into (4.45) lead to

$$H^{(3)}(Q, M_{pq}^{\gamma TM}) = \frac{4ja}{\omega \mu k_{pq} \sqrt{bc}} \left\{ \frac{k^2 q}{c} \sum_{r=0}^{\infty} \sum_{s=1}^{\infty} \frac{\epsilon_r (k_{rs}^{TM})^2 J'_r(k_{rs}^{TM} a) H^{\gamma TM \phi}}{2\gamma_{rs}^{TM} x_{rs}^2 J_{r+1}^2(x_{rs})} \right. \\ \left. - \frac{q}{c} \sum_{r=1}^{\infty} \sum_{s=1}^{\infty} \frac{\gamma_{rs}^{TE} (k_{rs}^{TE})^2 r J_r(k_{rs}^{TE} a) H^{\gamma TE \phi}}{(k_{rs}^{TE} a)(x_{rs}'^2 - r^2) J_r^2(x_{rs}') } \right. \\ \left. + (-1)^\gamma \left(\frac{p}{b}\right) \left(\frac{\sin \phi_0}{\phi_0}\right) \sum_{r=0}^{\infty} \sum_{s=1}^{\infty} \frac{\epsilon_r (k_{rs}^{TE})^3 J_r(k_{rs}^{TE} a) H^{\gamma TE z}}{2(x_{rs}'^2 - r^2) J_r^2(x_{rs}') } \right\} \quad (4.62)$$

where

$$H^{\gamma TM \phi} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\left(\frac{q\pi z'^+}{c}\right) \left\{ \frac{H^{\gamma TM \phi+}}{H^{\gamma TM \phi-}}, z' > z \right\} dz' \quad (4.63)$$

$$H^{\gamma TE \phi} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\left(\frac{q\pi z'^+}{c}\right) \left\{ \frac{H^{\gamma TE \phi+}}{H^{\gamma TE \phi-}}, z' > z \right\} dz' \quad (4.64)$$

$$H^{\gamma TE z} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin\left(\frac{q\pi z'^+}{c}\right) \left[H^{\gamma TE z \pm} + \left\{ \frac{H^{\gamma TE z+}}{H^{\gamma TE z-}}, z' > z \right\} \right] dz' \quad (4.65)$$

In (4.63)-(4.65),

$$H^{\gamma TM \phi-} = e^{-\gamma_{rs}^{TM} (L_3 - z')} \cosh(\gamma_{rs}^{TM} (L_3 - z)) \\ \cdot \left\{ \underline{u}_\rho \frac{\phi^{\gamma 1} r J_r(k_{rs}^{TM} \rho)}{k_{rs}^{TM} \rho} + \underline{u}_\phi \phi^{\gamma 2} J'_r(k_{rs}^{TM} \rho) \right\} \quad (4.66)$$

$$H^{\gamma TM \phi+} = \cosh(\gamma_{rs}^{TM} (L_3 - z')) e^{-\gamma_{rs}^{TM} (L_3 - z)} \\ \cdot \left\{ \underline{u}_\rho \frac{\phi^{\gamma 1} r J_r(k_{rs}^{TM} \rho)}{k_{rs}^{TM} \rho} + \underline{u}_\phi \phi^{\gamma 2} J'_r(k_{rs}^{TM} \rho) \right\} \quad (4.67)$$

$$H^{\gamma TE \phi -} = e^{-\gamma_{rs}^{TE}(L_3 - z')} \left\{ (\underline{u}_\rho \phi^{\gamma 1} J'_r(k_{rs}^{TE} \rho) + \underline{u}_\phi \frac{\phi^{\gamma 2} r J_r(k_{rs}^{TE} \rho)}{k_{rs}^{TE} \rho}) \cdot \cosh(\gamma_{rs}^{TE}(L_3 - z)) - \underline{u}_z \frac{\phi^{\gamma 1} k_{rs}^{TE} J_r(k_{rs}^{TE} \rho) \sinh(\gamma_{rs}^{TE}(L_3 - z))}{\gamma_{rs}^{TE}} \right\} \quad (4.68)$$

$$H^{\gamma TE \phi +} = \cosh(\gamma_{rs}^{TE}(L_3 - z')) e^{-\gamma_{rs}^{TE}(L_3 - z)} \cdot \left\{ \underline{u}_\rho \phi^{\gamma 1} J'_r(k_{rs}^{TE} \rho) + \underline{u}_\phi \frac{\phi^{\gamma 2} r J_r(k_{rs}^{TE} \rho)}{k_{rs}^{TE} \rho} + \underline{u}_z \frac{\phi^{\gamma 1} k_{rs}^{TE} J_r(k_{rs}^{TE} \rho)}{\gamma_{rs}^{TE}} \right\} \quad (4.69)$$

$$H^{\gamma TE z \pm} = \underline{u}_z \delta(z - z') \frac{\phi^{\gamma 4} J_r(k_{rs}^{TE} \rho)}{k_{rs}^{TE}} \quad (4.70)$$

$$H^{\gamma TE z -} = e^{-\gamma_{rs}^{TE}(L_3 - z')} \left\{ (\underline{u}_\rho \phi^{\gamma 4} J'_r(k_{rs}^{TE} \rho) - \underline{u}_\phi \frac{\phi^{\gamma 3} r J_r(k_{rs}^{TE} \rho)}{k_{rs}^{TE} \rho}) \cdot \cosh(\gamma_{rs}^{TE}(L_3 - z)) - \underline{u}_z \frac{\phi^{\gamma 4} k_{rs}^{TE} J_r(k_{rs}^{TE} \rho) \sinh(\gamma_{rs}^{TE}(L_3 - z))}{\gamma_{rs}^{TE}} \right\} \quad (4.71)$$

$$H^{\gamma TE z +} = -\sinh(\gamma_{rs}^{TE}(L_3 - z')) e^{-\gamma_{rs}^{TE}(L_3 - z)} \cdot \left\{ \underline{u}_\rho \phi^{\gamma 4} J'_r(k_{rs}^{TE} \rho) - \underline{u}_\phi \frac{\phi^{\gamma 3} r J_r(k_{rs}^{TE} \rho)}{k_{rs}^{TE} \rho} + \underline{u}_z \frac{\phi^{\gamma 4} k_{rs}^{TE} J_r(k_{rs}^{TE} \rho)}{\gamma_{rs}^{TE}} \right\} \quad (4.72)$$

Replacing (n, p) in (4.16), (4.17), and (4.28) by (r, s) and then substituting the resulting expressions into (4.46), we obtain

$$H^{(3)}(0, M_{pq}^{\gamma TE}) = \frac{4ja}{\omega \mu k_{pq}} \sqrt{\epsilon_p \epsilon_q} \left\{ \frac{k^2 p}{b} \sum_{r=0}^{\infty} \sum_{s=1}^{\infty} \frac{\epsilon_r (k_{rs}^{TM})^2 J'_r(k_{rs}^{TM} a) H^{\gamma TM \phi}}{2 \gamma_{rs}^{TM} x_{rs}^2 J_{r+1}^2(x_{rs})} - \frac{p}{b} \sum_{r=1}^{\infty} \sum_{s=1}^{\infty} \frac{\gamma_{rs}^{TE} (k_{rs}^{TE})^2 r J_r(k_{rs}^{TE} a) H^{\gamma TE \phi}}{(k_{rs}^{TE} a) (x_{rs}'^2 - r^2) J_r^2(x_{rs}') } - (-1)^r \left(\frac{q}{c} \right) \left(\frac{\sin \phi_o}{\phi_o} \right) \sum_{r=0}^{\infty} \sum_{s=1}^{\infty} \frac{\epsilon_r (k_{rs}^{TE})^3 J_r(k_{rs}^{TE} a) H^{\gamma TE z}}{2 (x_{rs}'^2 - r^2) J_r^2(x_{rs}') } \right\} \quad (4.73)$$

Substitution of (4.51)–(4.56) into (4.48)–(4.50) gives

$$\underline{E}^{\gamma TM \phi} = \left\{ -z^{TM1} \sinh(\gamma_{rs}^{TM}(L_3 - z)) + z^{TM2} e^{-\gamma_{rs}^{TM}(L_3 - z)} \right\} \cdot \left\{ \underline{u}_\rho \phi^{\gamma 2} J'_r(k_{rs}^{TM} \rho) - \underline{u}_\phi \frac{\phi^{\gamma 1} r J_r(k_{rs}^{TM} \rho)}{k_{rs}^{TM} \rho} \right\} + \left\{ z^{TM1} \cosh(\gamma_{rs}^{TM}(L_3 - z)) + z^{TM2} e^{-\gamma_{rs}^{TM}(L_3 - z)} \right\} \underline{u}_z \frac{\phi^{\gamma 2} k_{rs}^{TM} J_r(k_{rs}^{TM} \rho)}{\gamma_{rs}^{TM}} \quad (4.74)$$

$$\underline{E}^{\gamma TE\phi} = \left\{ -z^{TE1} \sinh(\gamma_{rs}^{TE}(L_3 - z)) + z^{TE2} e^{-\gamma_{rs}^{TE}(L_3 - z)} \right\} \cdot \left\{ \underline{u}_\rho \frac{\phi^{\gamma 2} r J_r(k_{rs}^{TE} \rho)}{k_{rs}^{TE} \rho} - \underline{u}_\phi \phi^{\gamma 1} J'_r(k_{rs}^{TE} \rho) \right\} \quad (4.75)$$

$$\underline{E}^{\gamma TEz} = \left\{ z^{TE3} \sinh(\gamma_{rs}^{TE}(L_3 - z)) + z^{TE4} e^{-\gamma_{rs}^{TE}(L_3 - z)} \right\} \cdot (-1)^\gamma \left\{ \underline{u}_\rho \frac{\phi^{\gamma 3} r J_r(k_{rs}^{TE} \rho)}{k_{rs}^{TE} \rho} + \underline{u}_\phi \phi^{\gamma 4} J'_r(k_{rs}^{TE} \rho) \right\} \quad (4.76)$$

where, for δ equal to TM or TE ,

$$z^{\delta 1} = 0, \quad z < -\frac{c}{2} \quad (4.77)$$

$$z^{\delta 1} = \int_{-\frac{c}{2}}^z e^{-\gamma_{rs}^\delta(L_3 - z')} \cos\left(\frac{q\pi z'^+}{c}\right) dz', \quad -\frac{c}{2} \leq z \leq \frac{c}{2} \quad (4.78)$$

$$z^{\delta 1} = \int_{-\frac{c}{2}}^{\frac{c}{2}} e^{-\gamma_{rs}^\delta(L_3 - z')} \cos\left(\frac{q\pi z'^+}{c}\right) dz', \quad z > \frac{c}{2} \quad (4.79)$$

$$z^{\delta 2} = \int_{-\frac{c}{2}}^{\frac{c}{2}} \cosh(\gamma_{rs}^\delta(L_3 - z')) \cos\left(\frac{q\pi z'^+}{c}\right) dz', \quad z < -\frac{c}{2} \quad (4.80)$$

$$z^{\delta 2} = \int_z^{\frac{c}{2}} \cosh(\gamma_{rs}^\delta(L_3 - z')) \cos\left(\frac{q\pi z'^+}{c}\right) dz', \quad -\frac{c}{2} \leq z \leq \frac{c}{2} \quad (4.81)$$

$$z^{\delta 2} = 0, \quad z > \frac{c}{2} \quad (4.82)$$

and

$$z^{TE3} = 0, \quad z < -\frac{c}{2} \quad (4.83)$$

$$z^{TE3} = \int_{-\frac{c}{2}}^z e^{-\gamma_{rs}^{TE}(L_3 - z')} \sin\left(\frac{q\pi z'^+}{c}\right) dz', \quad -\frac{c}{2} \leq z \leq \frac{c}{2} \quad (4.84)$$

$$z^{TE3} = \int_{-\frac{c}{2}}^{\frac{c}{2}} e^{-\gamma_{rs}^{TE}(L_3 - z')} \sin\left(\frac{q\pi z'^+}{c}\right) dz', \quad z > \frac{c}{2} \quad (4.85)$$

$$z^{TE4} = \int_{-\frac{c}{2}}^{\frac{c}{2}} \sinh(\gamma_{rs}^{TE}(L_3 - z')) \sin\left(\frac{q\pi z'^+}{c}\right) dz', \quad z < -\frac{c}{2} \quad (4.86)$$

$$z^{TE4} = \int_z^{\frac{c}{2}} \sinh(\gamma_{rs}^{TE}(L_3 - z')) \sin\left(\frac{q\pi z'^+}{c}\right) dz', \quad -\frac{c}{2} \leq z \leq \frac{c}{2} \quad (4.87)$$

$$z^{TE4} = 0, \quad z > \frac{c}{2} \quad (4.88)$$

Substitution of (4.66)–(4.72) into (4.63)–(4.65) gives

$$\underline{H}^{\gamma TM\phi} = \left\{ z^{TM1} \cosh(\gamma_{rs}^{TM}(L_3 - z)) + z^{TM2} e^{-\gamma_{rs}^{TM}(L_3 - z)} \right\}$$

$$\cdot \left\{ \underline{u}_\rho \frac{\phi^{\gamma 1} r J_r(k_{rs}^{TM} \rho)}{k_{rs}^{TM} \rho} + \underline{u}_\phi \phi^{\gamma 2} J'_r(k_{rs}^{TM} \rho) \right\} \quad (4.89)$$

$$\begin{aligned} H^{\gamma TE\phi} = & \left\{ z^{TE1} \cosh(\gamma_{rs}^{TE}(L_3 - z)) + z^{TE2} e^{-\gamma_{rs}^{TE}(L_3 - z)} \right\} \\ & \cdot \left\{ \underline{u}_\rho \phi^{\gamma 1} J'_r(k_{rs}^{TE} \rho) + \underline{u}_\phi \frac{\phi^{\gamma 2} r J_r(k_{rs}^{TE} \rho)}{k_{rs}^{TE} \rho} \right\} \\ & + \left\{ -z^{TE1} \sinh(\gamma_{rs}^{TE}(L_3 - z)) + z^{TE2} e^{-\gamma_{rs}^{TE}(L_3 - z)} \right\} \underline{u}_z \frac{\phi^{\gamma 1} k_{rs}^{TE} J_r(k_{rs}^{TE} \rho)}{\gamma_{rs}^{TE}} \end{aligned} \quad (4.90)$$

$$\begin{aligned} H^{\gamma TEz} = & \left\{ z^{TE3} \cosh(\gamma_{rs}^{TE}(L_3 - z)) - z^{TE4} e^{-\gamma_{rs}^{TE}(L_3 - z)} \right\} \\ & \cdot \left\{ \underline{u}_\rho \phi^{\gamma 4} J'_r(k_{rs}^{TE} \rho) - \underline{u}_\phi \frac{\phi^{\gamma 3} r J_r(k_{rs}^{TE} \rho)}{k_{rs}^{TE} \rho} \right\} + \left\{ z^{TE5} \right. \\ & \left. - z^{TE3} \sinh(\gamma_{rs}^{TE}(L_3 - z)) - z^{TE4} e^{-\gamma_{rs}^{TE}(L_3 - z)} \right\} \underline{u}_z \frac{\phi^{\gamma 4} k_{rs}^{TE} J_r(k_{rs}^{TE} \rho)}{\gamma_{rs}^{TE}} \end{aligned} \quad (4.91)$$

where

$$z^{TE5} = \begin{cases} \frac{\gamma_{rs}^{TE} \sin(\frac{\pi z^+}{c})}{(k_{rs}^{TE})^2}, & -\frac{\epsilon}{2} \leq z \leq \frac{\epsilon}{2} \\ 0, & \text{otherwise} \end{cases} \quad (4.92)$$

In (4.92), z^+ is given by (2.17).

So far in Chapter 4, we have found that $\underline{E}^{(3)}(0, \underline{M}_{pq}^{\gamma TM})$, $\underline{E}^{(3)}(0, \underline{M}_{pq}^{\gamma TE})$, $\underline{H}^{(3)}(0, \underline{M}_{pq}^{\gamma TM})$, and $\underline{H}^{(3)}(0, \underline{M}_{pq}^{\gamma TE})$ are given by (4.47), (4.61), (4.62), and (4.73), respectively. In (4.47) and (4.61), $\underline{E}^{\gamma TM\phi}$, $\underline{E}^{\gamma TE\phi}$, and $\underline{E}^{\gamma TEz}$ are given by (4.74), (4.75), and (4.76), respectively. In (4.62) and (4.73), $\underline{H}^{\gamma TM\phi}$, $\underline{H}^{\gamma TE\phi}$, and $\underline{H}^{\gamma TEz}$ are given by (4.89), (4.90), and (4.91), respectively.

The quantities $\phi^{\gamma 1}$, $\phi^{\gamma 2}$, $\phi^{\gamma 3}$, and $\phi^{\gamma 4}$ which appear in (4.74)–(4.76) and (4.89)–(4.91) are evaluated in Appendix E. These quantities are given by (E.10)–(E.13) in which $\phi_p^{(1)}$, $\phi_p^{(2)}$, $\phi_p^{(3)}$, and $\phi_p^{(4)}$ are given by (E.23)–(E.26).

The quantities z^{TM1} , z^{TM2} , z^{TE1} , z^{TE2} , z^{TE3} , and z^{TE4} which appear in (4.74)–(4.76) and (4.89)–(4.91) are evaluated for $(-\frac{\epsilon}{2} \leq z \leq \frac{\epsilon}{2})$ in Appendix F. As indicated in Table F.1, the quantities z^{TM1} and z^{TM2} are suitably given by (F.25) and (F.26) when γ_{rs}^{TM} is purely imaginary and by (F.32) and (F.33) when γ_{rs}^{TM} is purely real. In (F.25) and (F.26), β_{rs}^δ is given by (F.27), $q^{\delta-}$ is given by (F.23), and $q^{\delta+}$ by (F.24). The quantities z^{TE1} , z^{TE2} , z^{TE3} , and z^{TE4} are suitably given by (F.25), (F.26), (F.34), and (F.35) when

γ_{rs}^{TE} is purely imaginary and by (F.32), (F.33), (F.41), and (F.42) when γ_{rs}^{TE} is purely real.

If $z < -\frac{c}{2}$, then $z^{TM1} = z^{TE1} = z^{TE3} = 0$ while z^{TM2} , z^{TE2} , and z^{TE4} are given by their expressions in Appendix F with z^+ replaced by 0. If $z > \frac{c}{2}$, then $z^{TM2} = z^{TE2} = z^{TE4} = 0$ while z^{TM1} , z^{TE1} , and z^{TE3} are given by their expressions in Appendix F with z^+ replaced by c .

In view of (4.89)–(4.91), substitution of (4.31) and (4.62) into (2.30) gives

$$Y_{ij}^{3,\alpha TM, \gamma TM} = T \left\{ \frac{nq}{c^2} (S_1 - S_2) + \frac{mq}{bc} S_3 - \frac{np}{bc} S_4 - \frac{mp}{b^2} S_5 \right\} \quad (4.93)$$

where

$$T = \frac{8\pi j a^2}{k_{mn} k_{pq} \omega \mu b c} \quad (4.94)$$

$$S_1 = k^2 \sum_{r=0}^{\infty} \sum_{s=1}^{\infty} \frac{\epsilon_r (k_{rs}^{TM})^2 J_r'^2 (k_{rs}^{TM} a) z^{(1)} \phi^{\alpha \gamma 2}}{2 \gamma_{rs}^{TM} x_{rs}^2 J_{r+1}^2 (x_{rs})} \quad (4.95)$$

$$S_2 = \sum_{r=1}^{\infty} \sum_{s=1}^{\infty} \frac{\gamma_{rs}^{TE} (k_{rs}^{TE})^2 r^2 J_r^2 (k_{rs}^{TE} a) z^{(2)} \phi^{\alpha \gamma 2}}{(k_{rs}^{TE} a)^2 (x_{rs}'^2 - r^2) J_r^2 (x_{rs}')} \quad (4.96)$$

$$S_3 = (-1)^\alpha \left(\frac{\sin \phi_o}{\phi_o} \right) \sum_{r=1}^{\infty} \sum_{s=1}^{\infty} \frac{(k_{rs}^{TE})^3 r J_r^2 (k_{rs}^{TE} a) z^{(3)} \phi^{\alpha \gamma 1}}{(k_{rs}^{TE} a) (x_{rs}'^2 - r^2) J_r^2 (x_{rs}')} \quad (4.97)$$

$$S_4 = (-1)^\gamma \left(\frac{\sin \phi_o}{\phi_o} \right) \sum_{r=0}^{\infty} \sum_{s=1}^{\infty} \frac{\epsilon_r (k_{rs}^{TE})^3 r J_r^2 (k_{rs}^{TE} a) z^{(4)} \phi^{\alpha \gamma 3}}{2 (k_{rs}^{TE} a) (x_{rs}'^2 - r^2) J_r^2 (x_{rs}')} \quad (4.98)$$

$$S_5 = (-1)^{\alpha+\gamma} \left(\frac{\sin \phi_o}{\phi_o} \right)^2 \sum_{r=0}^{\infty} \sum_{s=1}^{\infty} \frac{\epsilon_r (k_{rs}^{TE})^4 J_r^2 (k_{rs}^{TE} a) z^{(5)} \phi^{\alpha \gamma 4}}{2 \gamma_{rs}^{TE} (x_{rs}'^2 - r^2) J_r^2 (x_{rs}')} \quad (4.99)$$

In (4.95)–(4.99),

$$\phi^{\alpha \gamma 1} = \int_{\phi_3}^{\phi_4} \phi^{\gamma 1} \cos\left(\frac{m\pi y^{\alpha+}}{b}\right) d\phi \quad (4.100)$$

$$\phi^{\alpha \gamma 2} = \int_{\phi_3}^{\phi_4} \phi^{\gamma 2} \sin\left(\frac{m\pi y^{\alpha+}}{b}\right) d\phi \quad (4.101)$$

$$\phi^{\alpha \gamma 3} = \int_{\phi_3}^{\phi_4} \phi^{\gamma 3} \sin\left(\frac{m\pi y^{\alpha+}}{b}\right) d\phi \quad (4.102)$$

$$\phi^{\alpha \gamma 4} = \int_{\phi_3}^{\phi_4} \phi^{\gamma 4} \cos\left(\frac{m\pi y^{\alpha+}}{b}\right) d\phi \quad (4.103)$$

where

$$\phi_3 = (2 - \alpha)\pi - \phi_o \quad (4.104)$$

$$\phi_4 = (2 - \alpha)\pi + \phi_o \quad (4.105)$$

Still in (4.95)–(4.99),

$$z^{(1)} = \int_{-\frac{\epsilon}{2}}^{\frac{\epsilon}{2}} \left\{ z^{TM1} \cosh(\gamma_{rs}^{TM}(L_3 - z)) + z^{TM2} e^{-\gamma_{rs}^{TM}(L_3 - z)} \right\} \cdot \cos\left(\frac{n\pi z^+}{c}\right) dz \quad (4.106)$$

$$z^{(2)} = \int_{-\frac{\epsilon}{2}}^{\frac{\epsilon}{2}} \left\{ z^{TE1} \cosh(\gamma_{rs}^{TE}(L_3 - z)) + z^{TE2} e^{-\gamma_{rs}^{TE}(L_3 - z)} \right\} \cdot \cos\left(\frac{n\pi z^+}{c}\right) dz \quad (4.107)$$

$$z^{(3)} = \int_{-\frac{\epsilon}{2}}^{\frac{\epsilon}{2}} \left\{ -z^{TE1} \sinh(\gamma_{rs}^{TE}(L_3 - z)) + z^{TE2} e^{-\gamma_{rs}^{TE}(L_3 - z)} \right\} \cdot \sin\left(\frac{n\pi z^+}{c}\right) dz \quad (4.108)$$

$$z^{(4)} = \int_{-\frac{\epsilon}{2}}^{\frac{\epsilon}{2}} \left\{ z^{TE3} \cosh(\gamma_{rs}^{TE}(L_3 - z)) - z^{TE4} e^{-\gamma_{rs}^{TE}(L_3 - z)} \right\} \cdot \cos\left(\frac{n\pi z^+}{c}\right) dz \quad (4.109)$$

$$z^{(5)} = \int_{-\frac{\epsilon}{2}}^{\frac{\epsilon}{2}} \left\{ z^{TE5} - z^{TE3} \sinh(\gamma_{rs}^{TE}(L_3 - z)) - z^{TE4} e^{-\gamma_{rs}^{TE}(L_3 - z)} \right\} \cdot \sin\left(\frac{n\pi z^+}{c}\right) dz \quad (4.110)$$

In view of (4.89)–(4.91), substitution of (4.33) and (4.62) into (2.30) gives

$$Y_{ij}^{3,\alpha TE, \gamma TM} = T \sqrt{\frac{\epsilon_m \epsilon_n}{4}} \left\{ \frac{mq}{bc} (S_1 - S_2) - \frac{nq}{c^2} S_3 - \frac{mp}{b^2} S_4 + \frac{np}{bc} S_5 \right\} \quad (4.111)$$

In view of (4.89)–(4.91), substitution of (4.31) and (4.73) into (2.30) gives

$$Y_{ij}^{3,\alpha TM, \gamma TE} = T \sqrt{\frac{\epsilon_p \epsilon_q}{4}} \left\{ \frac{np}{bc} (S_1 - S_2) + \frac{mp}{b^2} S_3 + \frac{nq}{c^2} S_4 + \frac{mq}{bc} S_5 \right\} \quad (4.112)$$

Similar substitution of (4.33) and (4.73) into (2.30) gives

$$Y_{ij}^{3,\alpha TE,\gamma TE} = T \sqrt{\frac{\epsilon_m \epsilon_n \epsilon_p \epsilon_q}{16}} \left\{ \frac{mp}{b^2} (S_1 - S_2) - \frac{np}{bc} S_3 + \frac{mq}{bc} S_4 - \frac{nq}{c^2} S_5 \right\} \quad (4.113)$$

In the previous paragraph, we found $Y_{ij}^{3,\alpha TM,\gamma TM}$, $Y_{ij}^{3,\alpha TE,\gamma TM}$, $Y_{ij}^{3,\alpha TM,\gamma TE}$, and $Y_{ij}^{3,\alpha TE,\gamma TE}$ to be given by (4.93), (4.111), (4.112), and (4.113), respectively. In these equations, T , S_1 , S_2 , S_3 , S_4 , and S_5 are given by (4.95)–(4.99). The quantities $\phi^{\alpha\gamma 1}$, $\phi^{\alpha\gamma 2}$, $\phi^{\alpha\gamma 3}$, and $\phi^{\alpha\gamma 4}$ that appear in (4.95)–(4.99) are evaluated in Appendix E. These quantities are given by (E.31)–(E.34) in which $\phi_p^{(1)}$, $\phi_p^{(2)}$, $\phi_p^{(3)}$, and $\phi_p^{(4)}$ are given by (E.23)–(E.26). Also in (E.31)–(E.34), $\phi^{\alpha 1\gamma 1}$, $\phi^{\alpha 2\gamma 1}$, $\phi^{\alpha 1\gamma 2}$, and $\phi^{\alpha 2\gamma 2}$ are given by (E.46)–(E.49) and (E.53)–(E.56). The quantities $z^{(1)}$, $z^{(2)}$, $z^{(3)}$, $z^{(4)}$, and $z^{(5)}$ that appear in (4.95)–(4.99) are evaluated in Appendix F. These quantities are given by equations whose numbers are listed in Table F.1.

Chapter 5

The Excitation Vector

In this chapter, $I_i^{\alpha\beta}$ of (2.24) is evaluated. Approximate expressions for $\underline{H}^{(3)}(\underline{J}^{\text{imp}}, \underline{Q})$ and $\underline{M}_{mn}^{\alpha\beta}$ have to be found before the integral in (2.24) can be evaluated.

In (2.24), $\underline{H}^{(3)}(\underline{J}^{\text{imp}}, \underline{Q})$ is the magnetic field due to $\underline{J}^{\text{imp}}$ radiating in the circular waveguide with the apertures A_1 and A_2 closed by perfect conductors, with a perfectly conducting wall at $z = L_3$, and with a matched load at the other end. If the apertures were present, the only z traveling wave at $z = -\frac{\epsilon}{2}$ contained in the field of $\underline{J}^{\text{imp}}$ would be the unit amplitude z traveling TM_{01} wave. Since closing the apertures produces no z traveling waves in the region for which $z \leq -\frac{\epsilon}{2}$, the only z traveling wave contained in the field ($\underline{E}^{(3)}(\underline{J}^{\text{imp}}, \underline{Q}), \underline{H}^{(3)}(\underline{J}^{\text{imp}}, \underline{Q})$) at $z = -\frac{\epsilon}{2}$ is the unit amplitude z traveling TM_{01} wave. The field of this wave is ($\underline{E}_{01}^{TM\epsilon+}, \underline{H}_{01}^{TM\epsilon+}$) given by (B.1):

$$\underline{E}_{01}^{TM\epsilon+} = Z_{01}^{TM\epsilon o} \underline{e}_{01}^{TM\epsilon}(\rho, \phi) e^{-j\beta_{01}^{TM} z} + \underline{u}_z \frac{(k_{01}^{TM})^2 \psi_{01}^{TM\epsilon}(\rho, \phi) e^{-j\beta_{01}^{TM} z}}{j\omega\epsilon} \quad (5.1)$$

$$\underline{H}_{01}^{TM\epsilon+} = \underline{h}_{01}^{TM\epsilon}(\rho, \phi) e^{-j\beta_{01}^{TM} z} \quad (5.2)$$

Here, k_{01}^{TM} and $\psi_{01}^{TM\epsilon}(\rho, \phi)$ are given by (B.7), $\underline{e}_{01}^{TM\epsilon}(\rho, \phi)$ by (B.22), $\underline{h}_{01}^{TM\epsilon}(\rho, \phi)$ by (B.23), and $Z_{01}^{TM\epsilon o}$ by (B.25) in which γ_{01}^{TM} is $j\beta_{01}^{TM}$ where

$$\beta_{01}^{TM} = \sqrt{k^2 - (k_{01}^{TM})^2} \quad (5.3)$$

Now, β_{01}^{TM} is purely real because it was assumed that the TM_{01} mode propagates in the circular waveguide.

The field $(\underline{E}^{(3)}(\underline{J}^{\text{imp}}, \underline{Q}), \underline{H}^{(3)}(\underline{J}^{\text{imp}}, \underline{Q}))$ is $(\underline{E}_{01}^{TM\epsilon+}, \underline{H}_{01}^{TM\epsilon+})$ of (5.1) and (5.2) plus the reflection of $(\underline{E}_{01}^{TM\epsilon+}, \underline{H}_{01}^{TM\epsilon+})$ from the conducting wall at L_3 . Taking the reflected field proportional to $(\underline{E}_{01}^{TM\epsilon-}, \underline{H}_{01}^{TM\epsilon-})$ of (B.2) and requiring the ρ component of $\underline{E}^{(3)}(\underline{J}^{\text{imp}}, \underline{Q})$ to vanish at $z = L_3$, we obtain

$$\underline{E}^{(3)}(\underline{J}^{\text{imp}}, \underline{Q}) = \frac{2e^{-j\beta_{01}^{TM}L_3}}{\omega\epsilon} \left\{ \gamma_{01}^{TM} \sin(\beta_{01}^{TM}(L_3 - z)) \underline{e}_{01}^{TM\epsilon}(\rho, \phi) - \underline{u}_z j(k_{01}^{TM})^2 \psi_{01}^{TM\epsilon}(\rho, \phi) \cos(\beta_{01}^{TM}(L_3 - z)) \right\} \quad (5.4)$$

$$\underline{H}^{(3)}(\underline{J}^{\text{imp}}, \underline{Q}) = 2e^{-j\beta_{01}^{TM}L_3} \cos(\beta_{01}^{TM}(L_3 - z)) \underline{h}_{01}^{TM\epsilon}(\rho, \phi) \quad (5.5)$$

Equation (B.25) was used in obtaining (5.4). From (B.7), we have

$$\psi_{01}^{TM\epsilon}(\rho, \phi) = \frac{J_0(k_{01}^{TM}\rho)}{\sqrt{\pi}x_{01}J_1(x_{01})} \quad (5.6)$$

where

$$k_{01}^{TM} = \frac{x_{01}}{a} \quad (5.7)$$

Applying [4, eq. (D-15)] to (B.22) and (B.23), we obtain

$$\underline{e}_{01}^{TM\epsilon}(\rho, \phi) = \underline{u}_\rho \frac{J_1(k_{01}^{TM}\rho)}{\sqrt{\pi}aJ_1(x_{01})} \quad (5.8)$$

$$\underline{h}_{01}^{TM\epsilon}(\rho, \phi) = \underline{u}_\phi \frac{J_1(k_{01}^{TM}\rho)}{\sqrt{\pi}aJ_1(x_{01})} \quad (5.9)$$

Substitution of (5.6), (5.8), and (5.9) into (5.4) and (5.5) gives

$$\underline{E}^{(3)}(\underline{J}^{\text{imp}}, \underline{Q}) = \frac{2e^{-j\beta_{01}^{TM}L_3}}{\sqrt{\pi}a\omega\epsilon J_1(x_{01})} \left\{ \underline{u}_\rho \gamma_{01}^{TM} J_1(k_{01}^{TM}\rho) \sin(\beta_{01}^{TM}(L_3 - z)) - \underline{u}_z j k_{01}^{TM} J_0(k_{01}^{TM}\rho) \cos(\beta_{01}^{TM}(L_3 - z)) \right\} \quad (5.10)$$

$$\underline{H}^{(3)}(\underline{J}^{\text{imp}}, \underline{Q}) = \frac{2e^{-j\beta_{01}^{TM}L_3}}{\sqrt{\pi}aJ_1(x_{01})} \underline{u}_\phi J_1(k_{01}^{TM}\rho) \cos(\beta_{01}^{TM}(L_3 - z)) \quad (5.11)$$

Although not needed to evaluate $I_i^{\alpha\beta}$, expression (5.10) is needed to evaluate, as indicated in the last two sentences of Chapter 2, the field in the circular waveguide.

Replacing (γ, p, q) by (α, m, n) in (4.31)–(4.33), we have

$$\begin{aligned} M_{mn}^{\alpha TM}(\phi, z) = & -\frac{2\pi}{k_{mn}\sqrt{bc}} \left\{ \underline{u}_\phi \frac{n}{c} \sin\left(\frac{m\pi y^{\alpha+}}{b}\right) \cos\left(\frac{n\pi z^+}{c}\right) \right. \\ & \left. - \underline{u}_z (-1)^\alpha \left(\frac{m}{b}\right) \left(\frac{\sin \phi_o}{\phi_o}\right) \cos\left(\frac{m\pi y^{\alpha+}}{b}\right) \sin\left(\frac{n\pi z^+}{c}\right) \right\} \quad (5.12) \end{aligned}$$

$$\begin{aligned} M_{mn}^{\alpha TE}(\phi, z) = & -\frac{2\pi}{k_{mn}} \sqrt{\frac{\epsilon_m \epsilon_n}{4bc}} \left\{ \underline{u}_\phi \frac{m}{b} \sin\left(\frac{m\pi y^{\alpha+}}{b}\right) \cos\left(\frac{n\pi z^+}{c}\right) \right. \\ & \left. + \underline{u}_z (-1)^\alpha \left(\frac{n}{c}\right) \left(\frac{\sin \phi_o}{\phi_o}\right) \cos\left(\frac{m\pi y^{\alpha+}}{b}\right) \sin\left(\frac{n\pi z^+}{c}\right) \right\} \quad (5.13) \end{aligned}$$

where

$$k_{mn} = \sqrt{\left(\frac{m\pi}{b}\right)^2 + \left(\frac{n\pi}{c}\right)^2} \quad (5.14)$$

In (5.12) and (5.13), $y^{\alpha+}$ and z^+ are given by (2.15)–(2.17).

Substituting (5.11)–(5.13) into (2.24) and integrating with respect to $y^{\alpha+}$ and z^+ rather than ϕ and z , we arrive at

$$I_i^{\alpha TM} = \frac{8\phi_o n}{k_{mn} a} \sqrt{\frac{a^2 \pi}{bc}} y_{sm} z_{ccn} e^{-j\beta_{01}^{TM} L_3} \quad (5.15)$$

$$I_i^{\alpha TE} = \frac{8\phi_o m}{k_{mn} a} \left(\frac{c}{b}\right) \sqrt{\frac{a^2 \pi \epsilon_m \epsilon_n}{4bc}} y_{sm} z_{ccn} e^{-j\beta_{01}^{TM} L_3} \quad (5.16)$$

where

$$y_{sm} = \frac{1}{b} \int_0^b \sin\left(\frac{m\pi y^{\alpha+}}{b}\right) dy^{\alpha+} \quad (5.17)$$

$$z_{ccn} = \frac{1}{c} \int_0^c \cos(\beta_{01}^{TM} (L_3^+ - z^+)) \cos\left(\frac{n\pi z^+}{c}\right) dz^+ \quad (5.18)$$

In (5.18),

$$L_3^+ = L_3 + \frac{c}{2} \quad (5.19)$$

In obtaining (5.15) and (5.16), we used (5.7) and

$$x_o = \frac{b}{2\phi_o} \quad (5.20)$$

Equation (5.20) was found by substituting (1.2) into (2.18).

Evaluating the integral in (5.17), we have

$$y_{sm} = \begin{cases} 0, & m \text{ even} \\ \frac{2}{m\pi}, & m \text{ odd} \end{cases} \quad (5.21)$$

Expression (5.18) is expanded as [5, formula 401.06]

$$z_{ccn} = \frac{1}{2c} \left\{ \int_0^c \cos \left(\left(\frac{n\pi}{c} - \beta_{01}^{TM} \right) z^+ + \beta_{01}^{TM} L_3^+ \right) dz^+ + \int_0^c \cos \left(\left(\frac{n\pi}{c} + \beta_{01}^{TM} \right) z^+ - \beta_{01}^{TM} L_3^+ \right) dz^+ \right\} \quad (5.22)$$

Evaluating the integrals in (5.22), we obtain

$$z_{ccn} = \frac{1}{2} \left\{ \frac{\sin(n\pi - \beta_{01}^{TM} c + \beta_{01}^{TM} L_3^+) - \sin(\beta_{01}^{TM} L_3^+)}{n\pi - \beta_{01}^{TM} c} + \frac{\sin(n\pi + \beta_{01}^{TM} c - \beta_{01}^{TM} L_3^+) + \sin(\beta_{01}^{TM} L_3^+)}{n\pi + \beta_{01}^{TM} c} \right\} \quad (5.23)$$

If $n\pi \pm \beta_{01}^{TM} c = 0$, then the right-hand side of (5.23) must be replaced by its limit as $n\pi \pm \beta_{01}^{TM} c$ approaches zero. To render the value of this limit obvious and to avoid roundoff error when $|n\pi \pm \beta_{01}^{TM} c|$ is small, we recast (5.23) as

$$z_{ccn} = \frac{\sin(n\pi - \beta_{01}^{TM} c) \cos(\beta_{01}^{TM} L_3^+) - 2 \sin^2\left(\frac{n\pi - \beta_{01}^{TM} c}{2}\right) \sin(\beta_{01}^{TM} L_3^+)}{2(n\pi - \beta_{01}^{TM} c)} + \frac{\sin(n\pi + \beta_{01}^{TM} c) \cos(\beta_{01}^{TM} L_3^+) + 2 \sin^2\left(\frac{n\pi + \beta_{01}^{TM} c}{2}\right) \sin(\beta_{01}^{TM} L_3^+)}{2(n\pi + \beta_{01}^{TM} c)} \quad (5.24)$$

Now, $I_i^{\alpha TM}$ and $I_i^{\alpha TE}$ are given by (5.15) and (5.16) with y_{sm} and z_{ccn} given by (5.21) and (5.24).

Appendix A

Modes of the Rectangular Waveguide

Consider the rectangular waveguide whose cross section is shown in Fig. A.1. In Fig. A.1,

$$\left. \begin{aligned} y^+ &= y + \frac{b}{2} \\ z^+ &= z + \frac{c}{2} \end{aligned} \right\} \quad (\text{A.1})$$

where y and z are shown in Fig. 1.1. Four kinds of mode fields can exist in the waveguide of Fig. A.1. These mode fields are $(\underline{E}_{mn}^{TM+}, \underline{H}_{mn}^{TM+})$, $(\underline{E}_{mn}^{TM-}, \underline{H}_{mn}^{TM-})$, $(\underline{E}_{mn}^{TE+}, \underline{H}_{mn}^{TE+})$, and $(\underline{E}_{mn}^{TE-}, \underline{H}_{mn}^{TE-})$. Here, \underline{E} is the electric field and \underline{H} is the magnetic field. The superscript "TM" denotes transverse magnetic, "TE" denotes transverse electric, "+" indicates that the wave travels in the $+x$ direction, and "-" indicates that the wave travels in the $-x$ direction. Here, x is the rectangular coordinate measured in the $\underline{u}_y \times \underline{u}_z$ direction where \underline{u}_y and \underline{u}_z are the unit vectors in the y and z directions, respectively.

From the analysis in [4, sec. 8-1], we obtain

$$\left. \begin{aligned} \underline{E}_{mn}^{TM+} &= Z_{mn}^{TM} \underline{e}_{mn}^{TM}(y^+, z^+) e^{-\gamma_{mn}x} + \underline{u}_x \frac{k_{mn}^2 \psi_{mn}^{TM}(y^+, z^+) e^{-\gamma_{mn}x}}{j\omega\epsilon} \\ \underline{H}_{mn}^{TM+} &= \underline{h}_{mn}^{TM}(y^+, z^+) e^{-\gamma_{mn}x} \end{aligned} \right\} \quad (\text{A.2})$$

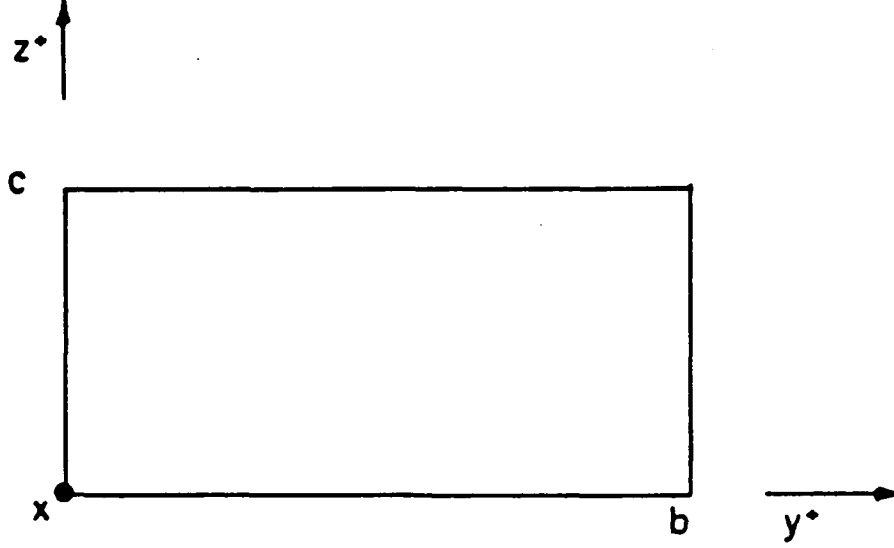


Figure A.1: Cross section of the rectangular waveguide.

and

$$\left. \begin{aligned} E_{mn}^{TM-} &= -Z_{mn}^{TM} \underline{e}_{mn}^{TM}(y^+, z^+) e^{\gamma_{mn} x} + \underline{u}_x \frac{k_{mn}^2 \psi_{mn}^{TM}(y^+, z^+) e^{\gamma_{mn} x}}{j\omega\epsilon} \\ H_{mn}^{TM-} &= \underline{h}_{mn}^{TM}(y^+, z^+) e^{\gamma_{mn} x} \end{aligned} \right\} \quad (\text{A.3})$$

where \underline{u}_x is the unit vector in the x direction. Moreover,

$$\underline{e}_{mn}^{TM}(y^+, z^+) = -\underline{\nabla} \psi_{mn}^{TM}(y^+, z^+) \quad (\text{A.4})$$

$$\underline{h}_{mn}^{TM}(y^+, z^+) = -\underline{u}_x \times \underline{\nabla} \psi_{mn}^{TM}(y^+, z^+) \quad (\text{A.5})$$

The wave function $\psi_{mn}^{TM}(y^+, z^+)$ satisfies

$$\nabla^2 \psi_{mn}^{TM} + k_{mn}^2 \psi_{mn}^{TM} = 0 \quad (\text{A.6})$$

subject to the boundary condition

$$\psi_{mn}^{TM} = 0 \quad (\text{A.7})$$

on the walls of the waveguide. Solutions to (A.6) and (A.7) are

$$\left. \begin{aligned} k_{mn}^2 &= \left(\frac{m\pi}{b}\right)^2 + \left(\frac{n\pi}{c}\right)^2 \\ \psi_{mn}^{TM}(y^+, z^+) &= \frac{2}{k_{mn}\sqrt{bc}} \sin\left(\frac{m\pi y^+}{b}\right) \sin\left(\frac{n\pi z^+}{c}\right) \end{aligned} \right\} \begin{aligned} m &= 1, 2, \dots \\ n &= 1, 2, \dots \end{aligned} \quad (\text{A.8})$$

The preceding ψ_{mn}^{TM} is normalized so that

$$\int_0^b dy^+ \int_0^c dz^+ \{ \nabla \psi_{mn}^{TM}(y^+, z^+) \} \cdot \{ \nabla \psi_{mn}^{TM}(y^+, z^+) \} = 1 \quad (\text{A.9})$$

Substituting ψ_{mn}^{TM} of (A.8) into (A.4) and (A.5) and taking ∇ with respect to the coordinates y^+ and z^+ , we obtain

$$\begin{aligned} \underline{e}_{mn}^{TM}(y^+, z^+) &= -\frac{2\pi}{k_{mn}\sqrt{bc}} \left\{ \underline{u}_y \frac{m}{b} \cos\left(\frac{m\pi y^+}{b}\right) \sin\left(\frac{n\pi z^+}{c}\right) \right. \\ &\quad \left. + \underline{u}_z \frac{n}{c} \sin\left(\frac{m\pi y^+}{b}\right) \cos\left(\frac{n\pi z^+}{c}\right) \right\} \end{aligned} \quad (\text{A.10})$$

$$\begin{aligned} \underline{h}_{mn}^{TM}(y^+, z^+) &= \frac{2\pi}{k_{mn}\sqrt{bc}} \left\{ \underline{u}_y \frac{n}{c} \sin\left(\frac{m\pi y^+}{b}\right) \cos\left(\frac{n\pi z^+}{c}\right) \right. \\ &\quad \left. - \underline{u}_z \frac{m}{b} \cos\left(\frac{m\pi y^+}{b}\right) \sin\left(\frac{n\pi z^+}{c}\right) \right\} \end{aligned} \quad (\text{A.11})$$

Remaining quantities in (A.2) and (A.3) are

$$\gamma_{mn} = \sqrt{k_{mn}^2 - k^2} \quad (\text{A.12})$$

and

$$Z_{mn}^{TM} = \frac{\gamma_{mn}}{j\omega\epsilon} \quad (\text{A.13})$$

In (A.12), $k = \omega\sqrt{\mu\epsilon}$ is assumed to be a real wave number. Here, ω is the angular frequency and μ and ϵ are, respectively, the permeability and permittivity of the homogeneous medium inside the waveguide. The radicand in (A.12) is therefore purely real so that γ_{mn} is either purely real or purely imaginary. If γ_{mn} is purely real, we take $\gamma_{mn} \geq 0$. If γ_{mn} is purely imaginary, we take the imaginary part of γ_{mn} to be non-negative.

From [4, sec. 8-1], we have

$$\left. \begin{aligned} E_{mn}^{TE+} &= e_{mn}^{TE}(y^+, z^+) e^{-\gamma_{mn}x} \\ H_{mn}^{TE+} &= Y_{mn}^{TE} h_{mn}^{TE}(y^+, z^+) e^{-\gamma_{mn}x} + u_x \frac{k_{mn}^2 \psi_{mn}^{TE}(y^+, z^+) e^{-\gamma_{mn}x}}{j\omega\mu} \end{aligned} \right\} \quad (\text{A.14})$$

and

$$\left. \begin{aligned} E_{mn}^{TE-} &= e_{mn}^{TE}(y^+, z^+) e^{\gamma_{mn}x} \\ H_{mn}^{TE-} &= -Y_{mn}^{TE} h_{mn}^{TE}(y^+, z^+) e^{\gamma_{mn}x} + u_x \frac{k_{mn}^2 \psi_{mn}^{TE}(y^+, z^+) e^{\gamma_{mn}x}}{j\omega\mu} \end{aligned} \right\} \quad (\text{A.15})$$

where

$$e_{mn}^{TE}(y^+, z^+) = u_x \times \nabla \psi_{mn}^{TE}(y^+, z^+) \quad (\text{A.16})$$

$$h_{mn}^{TE}(y^+, z^+) = -\nabla \psi_{mn}^{TE}(y^+, z^+) \quad (\text{A.17})$$

The wave function $\psi_{mn}^{TE}(y^+, z^+)$ satisfies

$$\nabla^2 \psi_{mn}^{TE} + k_{mn}^2 \psi_{mn}^{TE} = 0 \quad (\text{A.18})$$

subject to the boundary condition

$$u_n \cdot \nabla \psi_{mn}^{TE} = 0 \quad (\text{A.19})$$

on the walls of the waveguide. Here, u_n is the unit vector normal to the wall of the waveguide. Solutions to (A.18) and (A.19) are

$$\left. \begin{aligned} k_{mn}^2 &= \left(\frac{m\pi}{b}\right)^2 + \left(\frac{n\pi}{c}\right)^2 \\ \psi_{mn}^{TE}(y^+, z^+) &= \frac{1}{k_{mn}} \sqrt{\frac{\epsilon_m \epsilon_n}{bc}} \cos\left(\frac{m\pi y^+}{b}\right) \cos\left(\frac{n\pi z^+}{c}\right) \end{aligned} \right\} \begin{array}{l} m = 0, 1, 2, \dots \\ n = 0, 1, 2, \dots \\ m + n \neq 0 \end{array} \quad (\text{A.20})$$

where ϵ_n is Neumann's number given by

$$\epsilon_n = \begin{cases} 1, & n = 0 \\ 2, & n = 1, 2, \dots \end{cases} \quad (\text{A.21})$$

The preceding ψ_{mn}^{TE} is normalized so that

$$\int_0^b dy^+ \int_0^c dz^+ \left\{ \nabla \psi_{mn}^{TE}(y^+, z^+) \right\} \cdot \left\{ \nabla \psi_{mn}^{TE}(y^+, z^+) \right\} = 1 \quad (\text{A.22})$$

Substituting ψ_{mn}^{TE} of (A.20) into (A.16) and (A.17) and taking ∇ with respect to the coordinates y^+ and z^+ , we obtain

$$\begin{aligned} \underline{e}_{mn}^{TE}(y^+, z^+) = \frac{\pi}{k_{mn}} \sqrt{\frac{\epsilon_m \epsilon_n}{bc}} \left\{ \underline{u}_y \frac{n}{c} \cos\left(\frac{m\pi y^+}{b}\right) \sin\left(\frac{n\pi z^+}{c}\right) \right. \\ \left. - \underline{u}_z \frac{m}{b} \sin\left(\frac{m\pi y^+}{b}\right) \cos\left(\frac{n\pi z^+}{c}\right) \right\} \end{aligned} \quad (A.23)$$

$$\begin{aligned} \underline{h}_{mn}^{TE}(y^+, z^+) = \frac{\pi}{k_{mn}} \sqrt{\frac{\epsilon_m \epsilon_n}{bc}} \left\{ \underline{u}_y \frac{m}{b} \sin\left(\frac{m\pi y^+}{b}\right) \cos\left(\frac{n\pi z^+}{c}\right) \right. \\ \left. + \underline{u}_z \frac{n}{c} \cos\left(\frac{m\pi y^+}{b}\right) \sin\left(\frac{n\pi z^+}{c}\right) \right\} \end{aligned} \quad (A.24)$$

Remaining quantities in (A.14) and (A.15) are γ_{mn} given by (A.12) and

$$Y_{mn}^{TE} = \frac{\gamma_{mn}}{j\omega\mu} \quad (A.25)$$

Orthogonality relationships are

$$\begin{aligned} \int_0^b dy^+ \int_0^c dz^+ (\underline{e}_{mn}^\beta \cdot \underline{e}_{pq}^\delta) &= \int_0^b dy^+ \int_0^c dz^+ (\underline{h}_{mn}^\beta \cdot \underline{h}_{pq}^\delta) \\ &= \int_0^b dy^+ \int_0^c dz^+ (\underline{e}_{mn}^\beta \times \underline{h}_{pq}^\delta) \cdot \underline{u}_z = \begin{cases} 1, & (\beta, m, n) = (\delta, p, q) \\ 0, & \text{otherwise} \end{cases} \end{aligned} \quad (A.26)$$

where β is either *TM* or *TE* and δ is either *TM* or *TE*. Equality of the three integrals in (A.26) follows from (A.4), (A.5), (A.16), and (A.17). It is evident from [4, eq. (8-37)], (A.4), and (A.9) that the integrals are equal to the right-hand side of (A.26) when $\beta = \delta = TM$. It is evident from [4, eq. (8-36)], (A.17), and (A.22) that the integrals are equal to the right-hand side of (A.26) when $\beta = \delta = TE$. Because of [4, eq. (8-38)], the integrals vanish when $\beta \neq \delta$. If there is a degeneracy, that is, if $k_{mn} = k_{pq}$ for $(m, n) \neq (p, q)$, then, as pointed out in [4, p. 390], we must have

$$\int_0^b dy^+ \int_0^c dz^+ \left\{ \psi_{mn}^\beta(y^+, z^+) \psi_{pq}^\beta(y^+, z^+) \right\} = 0, \quad \beta = TE, TM \quad (A.27)$$

in order for (A.26) to hold. It can be shown that ψ_{mn}^{TM} of (A.8) and ψ_{mn}^{TE} of (A.20) satisfy (A.27).

In Appendix A, we found the four kinds of modes that can exist in the rectangular waveguide. There are two kinds of TM modes and two kinds of TE modes. The TM modes are $(E_{mn}^{TM+}, H_{mn}^{TM+})$ of (A.2) and $(E_{mn}^{TM-}, H_{mn}^{TM-})$ of (A.3). In (A.2) and (A.3), ψ_{mn}^{TM} , ϵ_{mn}^{TM} , and h_{mn}^{TM} are given by (A.8), (A.10), and (A.11), respectively. The TE modes are $(E_{mn}^{TE+}, H_{mn}^{TE+})$ of (A.14) and $(E_{mn}^{TE-}, H_{mn}^{TE-})$ of (A.15). In (A.14) and (A.15), ψ_{mn}^{TE} , ϵ_{mn}^{TE} , and h_{mn}^{TE} are given by (A.20), (A.23), and (A.24), respectively.

Appendix B

Modes of the Circular Waveguide

Eight kinds of modes can exist in the circular waveguide whose cross section is shown in Fig. B.1. They are $(E_{np}^{TMe+}, H_{np}^{TMe+})$, $(E_{np}^{TMe-}, H_{np}^{TMe-})$, $(E_{np}^{TMo+}, H_{np}^{TMo+})$, $(E_{np}^{TMo-}, H_{np}^{TMo-})$, $(E_{np}^{TEe+}, H_{np}^{TEe+})$, $(E_{np}^{TEe-}, H_{np}^{TEe-})$, $(E_{np}^{TEo+}, H_{np}^{TEo+})$, and $(E_{np}^{TEo-}, H_{np}^{TEo-})$. Here, E is the electric field and H is the magnetic field. The superscript "TM" denotes transverse magnetic, "TE" denotes transverse electric, "e" means even in ϕ , "o" means odd in ϕ , "+" indicates that the wave travels in the $+z$ direction, and "-" indicates that the wave travels in the $-z$ direction. Here, ϕ is the angle measured counterclockwise from the positive x axis and z is the cylindrical coordinate measured in the $\underline{u}_\rho \times \underline{u}_\phi$ direction where \underline{u}_ρ and \underline{u}_ϕ are the unit vectors in the ρ and ϕ directions, respectively. By definition, $\rho = \sqrt{x^2 + y^2}$.

From the analysis in [4, sec. 8-1], we obtain

$$\left. \begin{aligned} E_{np}^{TMe+} &= Z_{np}^{TMeo} \underline{e}_{np}^{TMe}(\rho, \phi) e^{-\gamma_{np}^{TM} z} + \underline{u}_z \frac{(k_{np}^{TM})^2 \psi_{np}^{TMe}(\rho, \phi) e^{-\gamma_{np}^{TM} z}}{j\omega\epsilon} \\ H_{np}^{TMe+} &= \underline{h}_{np}^{TMe}(\rho, \phi) e^{-\gamma_{np}^{TM} z} \end{aligned} \right\} \quad (B.1)$$

and

$$\left. \begin{aligned} E_{np}^{TMe-} &= -Z_{np}^{TMeo} \underline{e}_{np}^{TMe}(\rho, \phi) e^{\gamma_{np}^{TM} z} + \underline{u}_z \frac{(k_{np}^{TM})^2 \psi_{np}^{TMe}(\rho, \phi) e^{\gamma_{np}^{TM} z}}{j\omega\epsilon} \\ H_{np}^{TMe-} &= \underline{h}_{np}^{TMe}(\rho, \phi) e^{\gamma_{np}^{TM} z} \end{aligned} \right\} \quad (B.2)$$

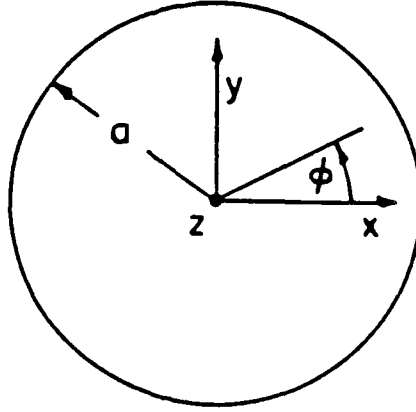


Figure B.1: Cross section of the circular waveguide.

where \underline{u}_z is the unit vector in the z direction. Moreover,

$$\underline{e}_{np}^{TM_e}(\rho, \phi) = -\underline{\nabla} \psi_{np}^{TM_e}(\rho, \phi) \quad (\text{B.3})$$

$$\underline{h}_{np}^{TM_e}(\rho, \phi) = -\underline{u}_z \times \underline{\nabla} \psi_{np}^{TM_e}(\rho, \phi) \quad (\text{B.4})$$

The wave function $\psi_{np}^{TM_e}(\rho, \phi)$ is a $\psi_{np}^{TM}(\rho, \phi)$ which is even in ϕ and which satisfies

$$\nabla^2 \psi_{np}^{TM}(\rho, \phi) + (k_{np}^{TM})^2 \psi_{np}^{TM}(\rho, \phi) = 0 \quad (\text{B.5})$$

subject to the boundary condition

$$\psi_{np}^{TM} = 0, \quad \rho = a \quad (\text{B.6})$$

Even solutions to (B.5) and (B.6) are

$$\left. \begin{aligned} k_{np}^{TM} &= \frac{x_{np}}{a} \\ \psi_{np}^{TM_e}(\rho, \phi) &= \sqrt{\frac{\epsilon_n}{\pi}} \frac{J_n(k_{np}^{TM} \rho) \cos(n\phi)}{x_{np} J_{n+1}(x_{np})} \end{aligned} \right\} \begin{aligned} n &= 0, 1, 2, \dots \\ p &= 1, 2, 3, \dots \end{aligned} \quad (\text{B.7})$$

where J_n is the Bessel function of the first kind of order n and $\{0 < x_{n1} < x_{n2} < x_{n3} \dots\}$ satisfies

$$J_n(x_{np}) = 0, \quad p = 1, 2, 3, \dots \quad (\text{B.8})$$

Furthermore, ϵ_n is Neumann's number:

$$\epsilon_n = \begin{cases} 1, & n = 0 \\ 2, & n = 1, 2, \dots \end{cases} \quad (\text{B.9})$$

The preceding $\psi_{np}^{TM\epsilon}$ is normalized so that

$$\int_0^a \rho d\rho \int_0^{2\pi} d\phi \left\{ \nabla \psi_{np}^{TM\epsilon}(\rho, \phi) \right\} \cdot \left\{ \nabla \psi_{np}^{TM\epsilon}(\rho, \phi) \right\} = 1 \quad (\text{B.10})$$

To verify (B.10), we first use (B.7) to obtain

$$\begin{aligned} \nabla \psi_{np}^{TM\epsilon}(\rho, \phi) = \sqrt{\frac{\epsilon_n}{\pi}} \left(\frac{1}{a J_{n+1}(x_{np})} \right) \left\{ \underline{u}_\rho J'_n(k_{np}^{TM} \rho) \cos(n\phi) \right. \\ \left. - \underline{u}_\phi \frac{n J_n(k_{np}^{TM} \rho) \sin(n\phi)}{k_{np}^{TM} \rho} \right\} \end{aligned} \quad (\text{B.11})$$

If the left-hand side of (B.10) is called I_1 , then substitution of (B.11) gives

$$\begin{aligned} I_1 = \frac{\epsilon_n}{\pi a^2 J_{n+1}^2(x_{np})} \int_0^a \rho d\rho \int_0^{2\pi} d\phi \\ \cdot \left\{ J_n'^2(k_{np}^{TM} \rho) \cos^2(n\phi) + \frac{n^2 J_n^2(k_{np}^{TM} \rho) \sin^2(n\phi)}{(k_{np}^{TM} \rho)^2} \right\} \end{aligned} \quad (\text{B.12})$$

Evaluation of the integral with respect to ϕ reduces (B.12) to

$$I_1 = \frac{2}{a^2 J_{n+1}^2(x_{np})} \int_0^a \rho d\rho \left\{ J_n'^2(k_{np}^{TM} \rho) + \frac{n^2 J_n^2(k_{np}^{TM} \rho)}{(k_{np}^{TM} \rho)^2} \right\} \quad (\text{B.13})$$

Substituting

$$x = k_{np}^{TM} \rho \quad (\text{B.14})$$

into (B.13), we obtain

$$I_1 = \frac{2I_2}{x_{np}^2 J_{n+1}^2(x_{np})} \quad (\text{B.15})$$

where

$$I_2 = \int_0^{x_{np}} \left\{ x J_n'^2(x) + \frac{n^2}{x} J_n^2(x) \right\} dx \quad (\text{B.16})$$

It is shown in Appendix C that

$$\begin{aligned} \int_0^d \left\{ x J_n'^2(x) + \frac{n^2}{x} J_n^2(x) \right\} dx &= \frac{1}{2} (d^2 - n^2) J_n^2(d) \\ &+ \frac{d^2}{2} J_n'^2(d) + d J_n(d) J_n'(d) \end{aligned} \quad (\text{B.17})$$

Replacing d by x_{np} in (B.17) and using (B.8), we obtain

$$I_2 = \frac{x_{np}^2 J_n'^2(x_{np})}{2} \quad (\text{B.18})$$

Now [6, formula 9.1.27],

$$J_n'(x) = -J_{n+1}(x) + \frac{n}{x} J_n(x) \quad (\text{B.19})$$

Replacing x by x_{np} in (B.19) and using (B.8), we obtain

$$J_n'(x_{np}) = -J_{n+1}(x_{np}) \quad (\text{B.20})$$

so that (B.18) becomes

$$I_2 = \frac{x_{np}^2 J_{n+1}^2(x_{np})}{2} \quad (\text{B.21})$$

Substitution of (B.21) into (B.15) gives $I_1 = 1$. Thus, (B.10) is verified.

Substitution of (B.11) into (B.3) and (B.4) gives

$$\begin{aligned} \underline{e}_{np}^{TM_e}(\rho, \phi) &= -\sqrt{\frac{\epsilon_n}{\pi}} \left(\frac{1}{a J_{n+1}(x_{np})} \right) \\ &\cdot \left\{ \underline{u}_\rho J_n'(k_{np}^{TM} \rho) \cos(n\phi) - \underline{u}_\phi \frac{n J_n(k_{np}^{TM} \rho) \sin(n\phi)}{k_{np}^{TM} \rho} \right\} \end{aligned} \quad (\text{B.22})$$

$$\begin{aligned} \underline{h}_{np}^{TM_e}(\rho, \phi) &= -\sqrt{\frac{\epsilon_n}{\pi}} \left(\frac{1}{a J_{n+1}(x_{np})} \right) \\ &\cdot \left\{ \underline{u}_\rho \frac{n J_n(k_{np}^{TM} \rho) \sin(n\phi)}{k_{np}^{TM} \rho} + \underline{u}_\phi J_n'(k_{np}^{TM} \rho) \cos(n\phi) \right\} \end{aligned} \quad (\text{B.23})$$

Remaining quantities in (B.1) and (B.2) are

$$\gamma_{np}^{TM} = \sqrt{(k_{np}^{TM})^2 - k^2} \quad (\text{B.24})$$

and

$$Z_{np}^{TMeo} = \frac{\gamma_{np}^{TM}}{j\omega\epsilon} \quad (\text{B.25})$$

In (B.24), $k = \omega\sqrt{\mu\epsilon}$ is assumed to be a real wave number. Here, ω is the angular frequency and μ and ϵ are, respectively, the permeability and the permittivity of the homogeneous medium inside the waveguide. The radicand in (B.24) is therefore purely real so that γ_{np}^{TM} is either purely real or purely imaginary. If γ_{np}^{TM} is purely real, we take $\gamma_{np}^{TM} \geq 0$. If γ_{np}^{TM} is purely imaginary, we take the imaginary part of γ_{np}^{TM} to be non-negative. The superscript "eo" was placed on the left-hand side of (B.25) to avoid confusion with the quantity Z_{mn}^{TM} which was defined in Appendix A.

Similar to (B.1)–(B.4), we have

$$\left. \begin{aligned} E_{np}^{TMo+} &= Z_{np}^{TMeo} \underline{e}_{np}^{TMo}(\rho, \phi) e^{-\gamma_{np}^{TM} z} + \underline{u}_z \frac{(k_{np}^{TM})^2 \psi_{np}^{TMo}(\rho, \phi) e^{-\gamma_{np}^{TM} z}}{j\omega\epsilon} \\ H_{np}^{TMo+} &= \underline{h}_{np}^{TMo}(\rho, \phi) e^{-\gamma_{np}^{TM} z} \end{aligned} \right\} \quad (\text{B.26})$$

and

$$\left. \begin{aligned} E_{np}^{TMo-} &= -Z_{np}^{TMeo} \underline{e}_{np}^{TMo}(\rho, \phi) e^{\gamma_{np}^{TM} z} + \underline{u}_z \frac{(k_{np}^{TM})^2 \psi_{np}^{TMo}(\rho, \phi) e^{\gamma_{np}^{TM} z}}{j\omega\epsilon} \\ H_{np}^{TMo-} &= \underline{h}_{np}^{TMo}(\rho, \phi) e^{\gamma_{np}^{TM} z} \end{aligned} \right\} \quad (\text{B.27})$$

where

$$\underline{e}_{np}^{TMo}(\rho, \phi) = -\nabla \psi_{np}^{TMo}(\rho, \phi) \quad (\text{B.28})$$

$$\underline{h}_{np}^{TMo}(\rho, \phi) = -\underline{u}_z \times \nabla \psi_{np}^{TMo}(\rho, \phi) \quad (\text{B.29})$$

The wave function $\psi_{np}^{TMo}(\rho, \phi)$ is a $\psi_{np}^{TM}(\rho, \phi)$ which is odd in ϕ and which satisfies (B.5) and (B.6). Odd solutions of (B.5) and (B.6) are

$$\left. \begin{aligned} k_{np}^{TM} &= \frac{x_{np}}{a} \\ \psi_{np}^{TMo}(\rho, \phi) &= \sqrt{\frac{2}{\pi}} \frac{J_n(k_{np}^{TM} \rho) \sin(n\phi)}{x_{np} J_{n+1}(x_{np})} \end{aligned} \right\} \begin{aligned} n &= 1, 2, 3, \dots \\ p &= 1, 2, 3, \dots \end{aligned} \quad (\text{B.30})$$

The preceding ψ_{np}^{TMo} is normalized so that

$$\int_0^a \rho d\rho \int_0^{2\pi} d\phi \{ \nabla \psi_{np}^{TMo}(\rho, \phi) \} \cdot \{ \nabla \psi_{np}^{TMo}(\rho, \phi) \} = 1 \quad (B.31)$$

To verify (B.31), we first use (B.30) to obtain

$$\begin{aligned} \nabla \psi_{np}^{TMo}(\rho, \phi) = & \sqrt{\frac{2}{\pi}} \left(\frac{1}{a J_{n+1}(x_{np})} \right) \\ & \cdot \left\{ \underline{u}_\rho J'_n(k_{np}^{TM} \rho) \sin(n\phi) + \underline{u}_\phi \frac{n J_n(k_{np}^{TM} \rho) \cos(n\phi)}{k_{np}^{TM} \rho} \right\} \end{aligned} \quad (B.32)$$

Substituting (B.32) for $\nabla \psi_{np}^{TMo}$, we find that the left-hand side of (B.31) is the right-hand side of (B.12) with ϵ_n replaced by 2 and with $\sin(n\phi)$ and $\cos(n\phi)$ interchanged. Integrating this result with respect to ϕ , we discover that the left-hand side of (B.31) is I_1 of (B.13). Now, as stated in the sentence following (B.21), $I_1 = 1$. Thus, (B.31) is verified.

Substitution of (B.32) into (B.28) and (B.29) gives

$$\begin{aligned} \underline{e}_{np}^{TMo}(\rho, \phi) = & -\sqrt{\frac{2}{\pi}} \left(\frac{1}{a J_{n+1}(x_{np})} \right) \\ & \cdot \left\{ \underline{u}_\rho J'_n(k_{np}^{TM} \rho) \sin(n\phi) + \underline{u}_\phi \frac{n J_n(k_{np}^{TM} \rho) \cos(n\phi)}{k_{np}^{TM} \rho} \right\} \end{aligned} \quad (B.33)$$

$$\begin{aligned} \underline{h}_{np}^{TMo}(\rho, \phi) = & \sqrt{\frac{2}{\pi}} \left(\frac{1}{a J_{n+1}(x_{np})} \right) \\ & \cdot \left\{ \underline{u}_\rho \frac{n J_n(k_{np}^{TM} \rho) \cos(n\phi)}{k_{np}^{TM} \rho} - \underline{u}_\phi J'_n(k_{np}^{TM} \rho) \sin(n\phi) \right\} \end{aligned} \quad (B.34)$$

The quantities γ_{np}^{TM} and Z_{np}^{TMeo} in (B.26) and (B.27) are given by (B.24) and (B.25), respectively.

From [4, sec. (8-1)], we have

$$\begin{aligned} \underline{E}_{np}^{TEe+} = & \underline{e}_{np}^{TEe}(\rho, \phi) e^{-\gamma_{np}^{TE} z} \\ \underline{H}_{np}^{TEe+} = & Y_{np}^{TEeo} \underline{h}_{np}^{TEe}(\rho, \phi) e^{-\gamma_{np}^{TE} z} + \underline{u}_z \frac{(k_{np}^{TE})^2 \psi_{np}^{TEe}(\rho, \phi) e^{-\gamma_{np}^{TE} z}}{j\omega\mu} \end{aligned} \quad (B.35)$$

and

$$\left. \begin{aligned} E_{np}^{TEe-} &= \underline{e}_{np}^{TEe}(\rho, \phi) e^{\gamma_{np}^{TE} z} \\ H_{np}^{TEe-} &= -Y_{np}^{TEe} \underline{h}_{np}^{TEe}(\rho, \phi) e^{\gamma_{np}^{TE} z} + \underline{u}_z \frac{(k_{np}^{TE})^2 \psi_{np}^{TEe}(\rho, \phi) e^{\gamma_{np}^{TE} z}}{j\omega\mu} \end{aligned} \right\} \quad (B.36)$$

where

$$\underline{e}_{np}^{TEe}(\rho, \phi) = \underline{u}_z \times \nabla \psi_{np}^{TEe}(\rho, \phi) \quad (B.37)$$

$$\underline{h}_{np}^{TEe}(\rho, \phi) = -\nabla \psi_{np}^{TEe}(\rho, \phi) \quad (B.38)$$

The wave function $\psi_{np}^{TEe}(\rho, \phi)$ is a $\psi_{np}^{TE}(\rho, \phi)$ which is even in ϕ and which satisfies

$$\nabla^2 \psi_{np}^{TE}(\rho, \phi) + (k_{np}^{TE})^2 \psi_{np}^{TE}(\rho, \phi) = 0 \quad (B.39)$$

subject to the boundary condition

$$\frac{\partial \psi_{np}^{TE}}{\partial \rho} = 0, \quad \rho = a \quad (B.40)$$

Even solutions of (B.39) and (B.40) are

$$\left. \begin{aligned} k_{np}^{TE} &= \frac{x'_{np}}{a} \\ \psi_{np}^{TEe}(\rho, \phi) &= \sqrt{\frac{\epsilon_n}{\pi(x_{np}'^2 - n^2)}} \frac{J_n(k_{np}^{TE} \rho) \cos(n\phi)}{J_n(x'_{np})} \end{aligned} \right\} \begin{aligned} n &= 0, 1, 2, \dots \\ p &= 1, 2, 3, \dots \end{aligned} \quad (B.41)$$

where $\{0 < x'_{n1} < x'_{n2} < x'_{n3} \dots\}$ satisfies

$$J'_n(x'_{np}) = 0, \quad p = 1, 2, 3, \dots \quad (B.42)$$

The preceding ψ_{np}^{TEe} is normalized so that

$$\int_0^a \rho d\rho \int_0^{2\pi} d\phi \left\{ \nabla \psi_{np}^{TEe}(\rho, \phi) \right\} \cdot \left\{ \nabla \psi_{np}^{TEe}(\rho, \phi) \right\} = 1 \quad (B.43)$$

To verify (B.43), we first use (B.41) to obtain

$$\begin{aligned} \nabla \psi_{np}^{TEe} &= \sqrt{\frac{\epsilon_n}{\pi(x_{np}'^2 - n^2)}} \left(\frac{k_{np}^{TE}}{J_n(x'_{np})} \right) \left\{ \underline{u}_\rho J'_n(k_{np}^{TE} \rho) \cos(n\phi) \right. \\ &\quad \left. - \underline{u}_\phi \frac{n J_n(k_{np}^{TE} \rho) \sin(n\phi)}{k_{np}^{TE} \rho} \right\} \end{aligned} \quad (B.44)$$

If the left-hand side of (B.43) is called I_3 , then substitution of (B.44) gives

$$I_3 = \frac{\epsilon_n}{\pi(x'_{np}{}^2 - n^2)} \left(\frac{k_{np}^{TE}}{J_n(x'_{np})} \right)^2 \int_0^a \rho d\rho \int_0^{2\pi} d\phi \cdot \left\{ J_n'^2(k_{np}^{TE} \rho) \cos^2(n\phi) + \frac{n^2 J_n^2(k_{np}^{TE} \rho) \sin^2(n\phi)}{(k_{np}^{TE} \rho)^2} \right\} \quad (B.45)$$

Evaluation of the integral with respect to ϕ reduces (B.45) to

$$I_3 = \frac{2}{x'_{np}{}^2 - n^2} \left(\frac{k_{np}^{TE}}{J_n(x'_{np})} \right)^2 \int_0^a \rho d\rho \left\{ J_n'^2(k_{np}^{TE} \rho) + \frac{n^2 J_n^2(k_{np}^{TE} \rho)}{(k_{np}^{TE} \rho)^2} \right\} \quad (B.46)$$

Substituting

$$x = k_{np}^{TE} \rho \quad (B.47)$$

into (B.46), we obtain

$$I_3 = \frac{2I_4}{(x'_{np}{}^2 - n^2) J_n^2(x'_{np})} \quad (B.48)$$

where

$$I_4 = \int_0^{x'_{np}} \left\{ x J_n'^2(x) + \frac{n^2 J_n^2(x)}{x} \right\} dx \quad (B.49)$$

Replacing d by x'_{np} in (B.17) and using (B.42), we obtain

$$I_4 = \frac{(x'_{np}{}^2 - n^2) J_n^2(x'_{np})}{2} \quad (B.50)$$

Substitution of (B.50) into (B.48) gives $I_3 = 1$. Thus, (B.43) is verified.

Substitution of (B.44) into (B.37) and (B.38) gives

$$\underline{e}_{np}^{TEe}(\rho, \phi) = \sqrt{\frac{\epsilon_n}{\pi(x'_{np}{}^2 - n^2)}} \left(\frac{k_{np}^{TE}}{J_n(x'_{np})} \right) \cdot \left\{ \underline{u}_\rho \frac{n J_n(k_{np}^{TE} \rho) \sin(n\phi)}{k_{np}^{TE} \rho} + \underline{u}_\phi J_n'(k_{np}^{TE} \rho) \cos(n\phi) \right\} \quad (B.51)$$

$$\underline{h}_{np}^{TEe}(\rho, \phi) = -\sqrt{\frac{\epsilon_n}{\pi(x'_{np}{}^2 - n^2)}} \left(\frac{k_{np}^{TE}}{J_n(x'_{np})} \right) \cdot \left\{ \underline{u}_\rho J_n'(k_{np}^{TE} \rho) \cos(n\phi) - \underline{u}_\phi \frac{n J_n(k_{np}^{TE} \rho) \sin(n\phi)}{k_{np}^{TE} \rho} \right\} \quad (B.52)$$

Remaining quantities in (B.35) and (B.36) are

$$\gamma_{np}^{TE} = \sqrt{(k_{np}^{TE})^2 - k^2} \quad (B.53)$$

and

$$Y_{np}^{TEeo} = \frac{\gamma_{np}^{TE}}{j\omega\mu} \quad (B.54)$$

The radicand in (B.53) is purely real so that γ_{np}^{TE} is either purely real or purely imaginary. If γ_{np}^{TE} is purely real, we take $\gamma_{np}^{TE} \geq 0$. If γ_{np}^{TE} is purely imaginary, we take the imaginary part of γ_{np}^{TE} to be non-negative. The superscript "eo" was placed on the left-hand side of (B.54) to avoid confusion with the quantity Y_{mn}^{TE} which was defined in Appendix A.

Similar to (B.35)–(B.38), we have

$$\left. \begin{aligned} E_{np}^{TEo+} &= e_{np}^{TEo}(\rho, \phi) e^{-\gamma_{np}^{TE} z} \\ H_{np}^{TEo+} &= Y_{np}^{TEeo} h_{np}^{TEo}(\rho, \phi) e^{-\gamma_{np}^{TE} z} + \underline{u}_z \frac{(k_{np}^{TE})^2 \psi_{np}^{TEo}(\rho, \phi) e^{-\gamma_{np}^{TE} z}}{j\omega\mu} \end{aligned} \right\} \quad (B.55)$$

and

$$\left. \begin{aligned} E_{np}^{TEo-} &= e_{np}^{TEo}(\rho, \phi) e^{\gamma_{np}^{TE} z} \\ H_{np}^{TEo-} &= -Y_{np}^{TEeo} h_{np}^{TEo}(\rho, \phi) e^{\gamma_{np}^{TE} z} + \underline{u}_z \frac{(k_{np}^{TE})^2 \psi_{np}^{TEo}(\rho, \phi) e^{\gamma_{np}^{TE} z}}{j\omega\mu} \end{aligned} \right\} \quad (B.56)$$

where

$$e_{np}^{TEo}(\rho, \phi) = \underline{u}_z \times \nabla \cdot \psi_{np}^{TEo}(\rho, \phi) \quad (B.57)$$

$$h_{np}^{TEo}(\rho, \phi) = -\nabla \psi_{np}^{TEo}(\rho, \phi) \quad (B.58)$$

The wave function $\psi_{np}^{TEo}(\rho, \phi)$ is a $\psi_{np}^{TE}(\rho, \phi)$ which is odd in ϕ and which satisfies (B.39) and (B.40). Odd solutions of (B.39) and (B.40) are

$$\left. \begin{aligned} k_{np}^{TE} &= \frac{x'_{np}}{a} \\ \psi_{np}^{TEo}(\rho, \phi) &= \sqrt{\frac{2}{\pi(x_{np}'^2 - n^2)}} \frac{J_n(k_{np}^{TE} \rho) \sin(n\phi)}{J_n(x_{np}') } \end{aligned} \right\} \begin{aligned} n &= 1, 2, 3, \dots \\ p &= 1, 2, 3, \dots \end{aligned} \quad (B.59)$$

The preceding ψ_{np}^{TEo} is normalized so that

$$\int_0^a \rho d\rho \int_0^{2\pi} d\phi \left\{ \nabla \psi_{np}^{TEo}(\rho, \phi) \right\} \cdot \left\{ \nabla \psi_{np}^{TEo}(\rho, \phi) \right\} = 1 \quad (\text{B.60})$$

To verify (B.60), we first use (B.59) to obtain

$$\begin{aligned} \nabla \psi_{np}^{TEo}(\rho, \phi) = & \sqrt{\frac{2}{\pi(x_{np}'^2 - n^2)}} \left(\frac{k_{np}^{TE}}{J_n(x_{np}')} \right) \\ & \cdot \left\{ \underline{u}_\rho J_n'(k_{np}^{TE} \rho) \sin(n\phi) + \underline{u}_\phi \frac{n J_n(k_{np}^{TE} \rho) \cos(n\phi)}{k_{np}^{TE} \rho} \right\} \quad (\text{B.61}) \end{aligned}$$

Substituting (B.61) for $\nabla \psi_{np}^{TEo}$, we find that the left-hand side of (B.60) is the right-hand side of (B.45) with ϵ_n replaced by 2 and with $\sin(n\phi)$ and $\cos(n\phi)$ interchanged. Integrating this result with respect to ϕ , we discover that the left-hand side of (B.60) is I_3 of (B.46). Now, as stated in the sentence following (B.50), $I_3 = 1$. Thus, (B.60) is verified.

Substitution of (B.61) into (B.57) and (B.58) gives

$$\begin{aligned} \underline{e}_{np}^{TEo}(\rho, \phi) = & -\sqrt{\frac{2}{\pi(x_{np}'^2 - n^2)}} \left(\frac{k_{np}^{TE}}{J_n(x_{np}')} \right) \\ & \cdot \left\{ \underline{u}_\rho \frac{n J_n(k_{np}^{TE} \rho) \cos(n\phi)}{k_{np}^{TE} \rho} - \underline{u}_\phi J_n'(k_{np}^{TE} \rho) \sin(n\phi) \right\} \quad (\text{B.62}) \end{aligned}$$

$$\begin{aligned} \underline{h}_{np}^{TEo}(\rho, \phi) = & -\sqrt{\frac{2}{\pi(x_{np}'^2 - n^2)}} \left(\frac{k_{np}^{TE}}{J_n(x_{np}')} \right) \\ & \cdot \left\{ \underline{u}_\rho J_n'(k_{np}^{TE} \rho) \sin(n\phi) + \underline{u}_\phi \frac{n J_n(k_{np}^{TE} \rho) \cos(n\phi)}{k_{np}^{TE} \rho} \right\} \quad (\text{B.63}) \end{aligned}$$

The quantities γ_{np}^{TE} and Y_{np}^{TEeo} in (B.55) and (B.56) are given by (B.53) and (B.54), respectively.

Orthogonality relationships are

$$\begin{aligned} & \int_0^a \rho d\rho \int_0^{2\pi} d\phi (\underline{e}_{mq}^{tr} \cdot \underline{e}_{np}^{us}) = \int_0^a \rho d\rho \int_0^{2\pi} d\phi (\underline{h}_{mq}^{tr} \cdot \underline{h}_{np}^{us}) \\ & = \int_0^a \rho d\rho \int_0^{2\pi} d\phi \left\{ (\underline{e}_{mq}^{tr} \times \underline{h}_{np}^{us}) \cdot \underline{u}_z \right\} = \begin{cases} 1, & (t, r, m, q) = (u, s, n, p) \\ 0, & \text{otherwise} \end{cases} \quad (\text{B.64}) \end{aligned}$$

where

$$\left. \begin{aligned} r &= e, o \\ s &= e, o \\ t &= TE, TM \\ u &= TE, TM \end{aligned} \right\} \quad (B.65)$$

Equality of the three integrals in (B.64) follows from (B.3), (B.4), (B.28), (B.29), (B.37), (B.38), (B.57), and (B.58). It is evident from [4, eq. (8-37)], (B.3), (B.10), (B.28), and (B.31) that the integrals are equal to the right-hand side of (B.64) when $t = u = TM$ and $r = s$. When $t = u = TM$ and $r \neq s$, the integrals vanish because the integrands are odd in ϕ . It is evident from [4, eq. (8-36)], (B.37), (B.43), (B.57), and (B.60) that the integrals are equal to the right-hand side of (B.64) when $t = u = TE$ and $r = s$. When $t = u = TE$ and $r \neq s$, the integrands vanish because the integrands are odd in ϕ . Because of [4, eq. (8-38)], the integrals vanish when $t \neq u$.

In Appendix B, we found the eight kinds of modes that can exist in the circular waveguide. There are two kinds of even TM modes, two kinds of odd TM modes, two kinds of even TE modes, and two kinds of odd TE modes. The two kinds of even TM modes are $(E_{np}^{TMe+}, H_{np}^{TMe+})$ of (B.1) and $(E_{np}^{TMe-}, H_{np}^{TMe-})$ of (B.2). In (B.1) and (B.2), ψ_{np}^{TMe} , e_{np}^{TMe} , and h_{np}^{TMe} are given by (B.7), (B.22), and (B.23), respectively. The two kinds of odd TM modes are $(E_{np}^{TMo+}, H_{np}^{TMo+})$ of (B.26) and $(E_{np}^{TMo-}, H_{np}^{TMo-})$ of (B.27). In (B.26), and (B.27), ψ_{np}^{TMo} , e_{np}^{TMo} , and h_{np}^{TMo} are given by (B.30), (B.33), and (B.34), respectively. The two kinds of even TE modes are $(E_{np}^{TEe+}, H_{np}^{TEe+})$ of (B.35) and $(E_{np}^{TEe-}, H_{np}^{TEe-})$ of (B.36). In (B.35) and (B.36), ψ_{np}^{TEe} , e_{np}^{TEe} , and h_{np}^{TEe} are given by (B.41), (B.51), and (B.52), respectively. The two kinds of odd TE modes are $(E_{np}^{TEo+}, H_{np}^{TEo+})$ of (B.55) and $(E_{np}^{TEo-}, H_{np}^{TEo-})$ of (B.56). In (B.55), and (B.56), ψ_{np}^{TEo} , e_{np}^{TEo} , and h_{np}^{TEo} are given by (B.59), (B.62), and (B.63), respectively.

Appendix C

Evaluation of an Integral of Bessel Functions

In Appendix C, we evaluate the integral I defined by

$$I = \int_0^d \left\{ x J_n'^2(x) + \frac{n^2}{x} J_n^2(x) \right\} dx \quad (\text{C.1})$$

Proceeding as in [7, sec. 5.296], we recast (C.1) as

$$I = \frac{1}{4} \int_0^d \left\{ (2J_n'(x))^2 + \left(\frac{2n}{x} J_n(x) \right)^2 \right\} x dx \quad (\text{C.2})$$

Substitution of [6, formula 9.1.27]

$$\frac{2n}{x} J_n(x) = J_{n-1}(x) + J_{n+1}(x) \quad (\text{C.3})$$

and

$$2J_n'(x) = J_{n-1}(x) - J_{n+1}(x) \quad (\text{C.4})$$

into (C.2) yields

$$I = \frac{1}{2} (I_{n-1} + I_{n+1}) \quad (\text{C.5})$$

where

$$I_n = \int_0^d x J_n^2(x) dx \quad (\text{C.6})$$

To evaluate I_n of (C.6), we will use Bessel's equation [6, formula 9.1.1]:

$$x^2 J_n''(x) + x J_n'(x) + (x^2 - n^2) J_n(x) = 0 \quad (C.7)$$

Multiplying (C.7) by $J_n'(x)$, we obtain

$$x^2 J_n'(x) J_n''(x) + x J_n'^2(x) + (x^2 - n^2) J_n(x) J_n'(x) = 0 \quad (C.8)$$

Expression (C.8) is recast as

$$\frac{x^2}{2} (J_n'^2(x))' + x J_n'^2(x) + \frac{x^2}{2} (J_n^2(x))' - \frac{n^2}{2} (J_n^2(x))' = 0 \quad (C.9)$$

The integral of (C.9) from 0 to d with respect to x is

$$\begin{aligned} \frac{1}{2} \int_0^d x^2 (J_n'^2(x))' dx + \int_0^d x J_n'^2(x) dx + \frac{1}{2} \int_0^d x^2 (J_n^2(x))' dx \\ - \frac{n^2}{2} (J_n^2(d) - J_n^2(0)) = 0 \end{aligned} \quad (C.10)$$

Because [6, formula 9.1.7]

$$J_n(0) = \begin{cases} 1, & n = 0 \\ 0, & n = 1, 2, 3, \dots \end{cases} \quad (C.11)$$

the $J_n^2(0)$ term drops out of (C.10). Integrating the first and third integrals in (C.10) by parts, we obtain

$$\frac{d^2}{2} (J_n'^2(d) + J_n^2(d)) - \int_0^d x J_n^2(x) dx - \frac{n^2}{2} J_n^2(d) = 0 \quad (C.12)$$

whence

$$I_n = \frac{1}{2} \{ d^2 J_n'^2(d) + (d^2 - n^2) J_n^2(d) \} \quad (C.13)$$

Substitution of (C.13) into (C.5) gives

$$\begin{aligned} I = \frac{1}{4} \{ (d^2 - (n-1)^2) J_{n-1}^2(d) + (d^2 - (n+1)^2) J_{n+1}^2(d) \\ + d^2 J_{n-1}'^2(d) + d^2 J_{n+1}'^2(d) \} \end{aligned} \quad (C.14)$$

Adding (C.4) to (C.3), we obtain

$$J_{n-1}(x) = \frac{n}{x}J_n(x) + J'_n(x) \quad (\text{C.15})$$

Subtracting (C.4) from (C.3), we obtain

$$J_{n+1}(x) = \frac{n}{x}J_n(x) - J'_n(x) \quad (\text{C.16})$$

From [6, formula 9.1.27], we have

$$J'_{n-1}(x) = -J_n(x) + \frac{n-1}{x}J_{n-1}(x) \quad (\text{C.17})$$

$$J'_{n+1}(x) = J_n(x) - \frac{n+1}{x}J_{n+1}(x) \quad (\text{C.18})$$

Substitution of (C.15) into (C.17) gives

$$J'_{n-1}(x) = \left(-1 + \frac{n(n-1)}{x^2}\right)J_n(x) + \frac{n-1}{x}J'_n(x) \quad (\text{C.19})$$

Substitution of (C.16) into (C.18) gives

$$J'_{n+1}(x) = \left(1 - \frac{n(n+1)}{x^2}\right)J_n(x) + \frac{n+1}{x}J'_n(x) \quad (\text{C.20})$$

Using (C.15), (C.16), (C.19), and (C.20) to convert the Bessel functions of order $n \pm 1$ in (C.14) to Bessel functions of order n , we arrive at

$$I = \frac{1}{2}(d^2 - n^2)J_n^2(d) + \frac{d^2}{2}J_n'^2(d) + dJ_n(d)J'_n(d) \quad (\text{C.21})$$

This result agrees with [7, eq. (12) on page 186].

Appendix D

Radiation of a Magnetic Current Element in a Circular Waveguide Matched at Both Ends

In Appendix D, we determine the fields $\underline{E}(\underline{Q}, \underline{u}_\phi \delta(\underline{r} - \underline{r}'))$, $\underline{H}(\underline{Q}, \underline{u}_\phi \delta(\underline{r} - \underline{r}'))$, $\underline{E}(\underline{Q}, \underline{u}_z \delta(\underline{r} - \underline{r}'))$, and $\underline{H}(\underline{Q}, \underline{u}_z \delta(\underline{r} - \underline{r}'))$ in the circular waveguide whose cross section is shown in Fig. B.1. For simplicity, we assume that the waveguide is matched at both ends. Here, $\underline{E}(\underline{J}^v, \underline{M}^v)$ is the electric field due to the combination of the electric current source \underline{J}^v and the magnetic current source \underline{M}^v where both \underline{J}^v and \underline{M}^v are volume densities of current. Similarly, $\underline{H}(\underline{J}^v, \underline{M}^v)$ is the magnetic field due to $(\underline{J}^v, \underline{M}^v)$. The argument " \underline{Q} " indicates that there is no electric current source in the waveguide. Furthermore, $\delta(\underline{r} - \underline{r}')$ is the three dimensional Dirac delta function, \underline{u}_ϕ is the unit vector in the ϕ direction, and \underline{u}_z is the unit vector in the z direction. Here, \underline{r} is the radius vector to the point whose cylindrical coordinates are (ρ, ϕ, z) , and \underline{r}' is the radius vector to the point whose cylindrical coordinates are (ρ', ϕ', z') .

Consider $\underline{E}(\underline{Q}, \underline{u}_\phi \delta(\underline{r} - \underline{r}'))$ and $\underline{H}(\underline{Q}, \underline{u}_\phi \delta(\underline{r} - \underline{r}'))$ evaluated at the point whose cylindrical coordinates are (ρ, ϕ, z) . Since the only source is a transverse magnetic current located at (ρ', ϕ', z') and since the waveguide is matched at both ends, there will only be waves that travel outward from z' and the transverse component of the magnetic field will be continuous at z' . Hence,

$E(\mathbf{Q}, \mathbf{u}_\phi \delta(\mathbf{r} - \mathbf{r}'))$ and $H(\mathbf{Q}, \mathbf{u}_\phi \delta(\mathbf{r} - \mathbf{r}'))$ may be expanded as

$$\begin{aligned} (E(\mathbf{Q}, \mathbf{u}_\phi \delta(\mathbf{r} - \mathbf{r}')), H(\mathbf{Q}, \mathbf{u}_\phi \delta(\mathbf{r} - \mathbf{r}'))) &= \sum_{n=0}^{\infty} \sum_{p=1}^{\infty} \sum_{s=e,o} \\ \{ C_{np}^{TM s, M\phi} e^{\pm \gamma_{np}^{TM} z'} (E_{np}^{TM s \pm}, H_{np}^{TM s \pm}) \pm C_{np}^{TE s, M\phi} e^{\pm \gamma_{np}^{TE} z'} (E_{np}^{TE s \pm}, H_{np}^{TE s \pm}) \} \end{aligned} \quad (D.1)$$

where the upper sign is to be taken for $z > z'$ and the lower one for $z < z'$. In (D.1), $(E_{np}^{TM s \pm}, H_{np}^{TM s \pm})$ and $(E_{np}^{TE s \pm}, H_{np}^{TE s \pm})$ are given by (B.1), (B.2), (B.26), (B.27), (B.35), (B.36), (B.55), and (B.56). Moreover, γ_{np}^{TM} and γ_{np}^{TE} are given by (B.24) and (B.53), respectively. The $s = o$ term is to be omitted from (D.1) when n , the index of the outer summation, is zero. In (D.1), $C_{np}^{TM s, M\phi}$ and $C_{np}^{TE s, M\phi}$ are unknown constants. The superscript " $M\phi$ " indicates that the source is the ϕ directed magnetic current $\mathbf{u}_\phi \delta(\mathbf{r} - \mathbf{r}')$. Substituting the expressions in Appendix B for $(E_{np}^{TM s \pm}, H_{np}^{TM s \pm})$ and $(E_{np}^{TE s \pm}, H_{np}^{TE s \pm})$ into (D.1), we obtain

$$\begin{aligned} E(\mathbf{Q}, \mathbf{u}_\phi \delta(\mathbf{r} - \mathbf{r}')) &= \sum_{n=0}^{\infty} \sum_{p=1}^{\infty} \sum_{s=e,o} C_{np}^{TM s, M\phi} \\ &\cdot \left\{ \epsilon(z, z') Z_{np}^{TM eo} \underline{e}_{np}^{TM s}(\rho, \phi) + \mathbf{u}_z \frac{(k_{np}^{TM})^2 \psi_{np}^{TM s}(\rho, \phi)}{j\omega\epsilon} \right\} e^{-\gamma_{np}^{TM} |z - z'|} \\ &+ \sum_{n=0}^{\infty} \sum_{p=1}^{\infty} \sum_{s=e,o} C_{np}^{TE s, M\phi} \epsilon(z, z') \underline{e}_{np}^{TE s}(\rho, \phi) e^{-\gamma_{np}^{TE} |z - z'|} \end{aligned} \quad (D.2)$$

$$\begin{aligned} H(\mathbf{Q}, \mathbf{u}_\phi \delta(\mathbf{r} - \mathbf{r}')) &= \sum_{n=0}^{\infty} \sum_{p=1}^{\infty} \sum_{s=e,o} C_{np}^{TM s, M\phi} \underline{h}_{np}^{TM s}(\rho, \phi) e^{-\gamma_{np}^{TM} |z - z'|} \\ &+ \sum_{n=0}^{\infty} \sum_{p=1}^{\infty} \sum_{s=e,o} C_{np}^{TE s, M\phi} \left\{ Y_{np}^{TE eo} \underline{h}_{np}^{TE s}(\rho, \phi) \right. \\ &\left. + \mathbf{u}_z \frac{\epsilon(z, z') (k_{np}^{TE})^2 \psi_{np}^{TE s}(\rho, \phi)}{j\omega\mu} \right\} e^{-\gamma_{np}^{TE} |z - z'|} \end{aligned} \quad (D.3)$$

where

$$\epsilon(z, z') = \begin{cases} 1, & z > z' \\ -1, & z < z' \end{cases} \quad (D.4)$$

Now,

$$\delta(\mathbf{r} - \mathbf{r}') = \delta(\rho - \rho') \delta(z - z') \quad (D.5)$$

where $\underline{\rho}$ is the radius vector to the point in the xy plane whose cylindrical coordinates are (ρ, ϕ) and $\underline{\rho}'$ is the radius vector to the point in the xy plane whose cylindrical coordinates are (ρ', ϕ') . Furthermore, $\delta(\underline{\rho} - \underline{\rho}')$ is the two-dimensional Dirac delta function, and $\delta(z - z')$ is the one-dimensional Dirac delta function. Thanks to (D.5), the fields on the left-hand sides of (D.2) and (D.3) become $\underline{E}(\underline{Q}, \underline{M}\delta(z - z'))$ and $\underline{H}(\underline{Q}, \underline{M}\delta(z - z'))$ where \underline{M} is a surface density of magnetic current in the $z = z'$ plane:

$$\underline{M} = \underline{u}_\phi \delta(\underline{\rho} - \underline{\rho}') \quad (\text{D.6})$$

The right-hand side of (D.2) is discontinuous at $z = z'$. The discontinuity is related to \underline{M} by [4, eq.(3.14)]

$$\underline{M} = (\underline{E}^+ - \underline{E}^-) \times \underline{u}_z \quad (\text{D.7})$$

where

$$\underline{E}^+ = \lim_{\substack{z \rightarrow z' \\ z > z'}} \underline{E}(\underline{Q}, \underline{M}\delta(z - z')) \quad (\text{D.8})$$

$$\underline{E}^- = \lim_{\substack{z \rightarrow z' \\ z < z'}} \underline{E}(\underline{Q}, \underline{M}\delta(z - z')) \quad (\text{D.9})$$

Taking $\underline{u}_z \times$ of both sides of (D.7), we obtain

$$\underline{u}_z \times \underline{M} = \underline{u}_z \times ((\underline{E}^+ - \underline{E}^-) \times \underline{u}_z) \quad (\text{D.10})$$

or, more simply,

$$\underline{u}_z \times \underline{M} = (\underline{E}^+ - \underline{E}^-)_{\text{tan}} \quad (\text{D.11})$$

where the subscript "tan" denotes the transverse component. Applying (D.11) to the discontinuous electric field of (D.2), we obtain

$$\begin{aligned} -\underline{u}_\rho \delta(\underline{\rho} - \underline{\rho}') &= 2 \sum_{n=0}^{\infty} \sum_{p=1}^{\infty} \sum_{s=e,o} C_{np}^{TM s, M\phi} Z_{np}^{TM eo} \underline{e}_{np}^{TM s}(\rho, \phi) \\ &+ 2 \sum_{n=0}^{\infty} \sum_{p=1}^{\infty} \sum_{s=e,o} C_{np}^{TE s, M\phi} \underline{e}_{np}^{TE s}(\rho, \phi) \end{aligned} \quad (\text{D.12})$$

Scalar multiplication of (D.12) by $\underline{e}_{mq}^{tr}(\rho, \phi)$ and integration over the cross section of the waveguide give

$$\begin{aligned}
& - \int_0^a \rho d\rho \int_0^{2\pi} d\phi \left\{ \underline{u}_\rho \cdot \underline{e}_{mq}^{tr}(\rho, \phi) \right\} \delta(\rho - \rho') \\
& = 2 \sum_{n=0}^{\infty} \sum_{p=1}^{\infty} \sum_{s=e,o} C_{np}^{TM_s, M\phi} Z_{np}^{TMeo} \int_0^a \rho d\rho \int_0^{2\pi} d\phi \left\{ \underline{e}_{mq}^{tr}(\rho, \phi) \cdot \underline{e}_{np}^{TM_s}(\rho, \phi) \right\} \\
& \quad + 2 \sum_{n=0}^{\infty} \sum_{p=1}^{\infty} \sum_{s=e,o} C_{np}^{TE_s, M\phi} \int_0^a \rho d\rho \int_0^{2\pi} d\phi \left\{ \underline{e}_{mq}^{tr}(\rho, \phi) \cdot \underline{e}_{np}^{TE_s}(\rho, \phi) \right\} \quad (D.13)
\end{aligned}$$

In (D.13), we choose

$$\left. \begin{aligned} r &= e, o \\ t &= TM, TE \\ m &= 0, 1, 2, \dots \\ q &= 1, 2, 3, \dots \end{aligned} \right\} \quad (D.14)$$

The definition of $\delta(\rho - \rho')$ and the orthogonality relationships (B.64) are used to evaluate the integrals in (D.13). Next, r , m , and q are replaced by s , n , and p , respectively. The result is

$$C_{np}^{TM_s, M\phi} = - \frac{\underline{u}_{\rho'} \cdot \underline{e}_{np}^{TM_s}(\rho', \phi')}{2Z_{np}^{TMeo}} \quad (D.15)$$

$$C_{np}^{TE_s, M\phi} = - \frac{1}{2} \underline{u}_{\rho'} \cdot \underline{e}_{np}^{TE_s}(\rho', \phi') \quad (D.16)$$

where $\underline{u}_{\rho'}$ is the unit vector in the ρ' direction.

Substitution of (D.15) and (D.16) into (D.2) and (D.3) gives

$$\begin{aligned}
E(\underline{Q}, \underline{u}_\phi \delta(\underline{r} - \underline{r}')) &= -\frac{1}{2} \sum_{n=0}^{\infty} \sum_{p=1}^{\infty} \sum_{s=e,o} \left(\underline{u}_{\rho'} \cdot \underline{e}_{np}^{TM_s}(\rho', \phi') \right) \\
& \quad \cdot \left\{ \epsilon(z, z') \underline{e}_{np}^{TM_s}(\rho, \phi) + \underline{u}_z \frac{(k_{np}^{TM})^2 \psi_{np}^{TM_s}(\rho, \phi)}{j\omega \epsilon Z_{np}^{TMeo}} \right\} e^{-\gamma_{np}^{TM} |z - z'|} \\
& \quad - \frac{1}{2} \sum_{n=0}^{\infty} \sum_{p=1}^{\infty} \sum_{s=e,o} \left(\underline{u}_{\rho'} \cdot \underline{e}_{np}^{TE_s}(\rho', \phi') \right) \epsilon(z, z') \underline{e}_{np}^{TE_s}(\rho, \phi) e^{-\gamma_{np}^{TE} |z - z'|} \quad (D.17)
\end{aligned}$$

$$H(\underline{Q}, \underline{u}_\phi \delta(\underline{r} - \underline{r}')) = -\frac{1}{2} \sum_{n=0}^{\infty} \sum_{p=1}^{\infty} \sum_{s=e,o} \frac{(\underline{u}_{\rho'} \cdot \underline{e}_{np}^{TM_s}(\rho', \phi'))}{Z_{np}^{TMeo}}$$

$$\begin{aligned} & \cdot \underline{h}_{np}^{TM s}(\rho, \phi) e^{-\gamma_{np}^{TM} |z-z'|} - \frac{1}{2} \sum_{n=0}^{\infty} \sum_{p=1}^{\infty} \sum_{s=e,o} (\underline{u}_{\rho'} \cdot \underline{e}_{np}^{TE s}(\rho', \phi')) \\ & \cdot \left\{ Y_{np}^{TE eo} \underline{h}_{np}^{TE s}(\rho, \phi) + \underline{u}_z \frac{\epsilon(z, z') (k_{np}^{TE})^2 \psi_{np}^{TE s}(\rho, \phi)}{j\omega\mu} \right\} e^{-\gamma_{np}^{TE} |z-z'|} \quad (D.18) \end{aligned}$$

Substituting (B.22), (B.33), (B.51), (B.62), (B.7), (B.30), and (B.25) into (D.17), we obtain

$$\begin{aligned} E(\underline{Q}, \underline{u}_{\phi} \delta(\underline{r} - \underline{r}')) &= \frac{1}{2\pi} \sum_{n=0}^{\infty} \sum_{p=1}^{\infty} \frac{\epsilon_n (k_{np}^{TM})^2 J'_n(k_{np}^{TM} \rho')}{x_{np}^2 J_{n+1}^2(x_{np})} \left\{ -\underline{u}_{\rho} \epsilon(z, z') \right. \\ & \cdot J'_n(k_{np}^{TM} \rho) \cos(n(\phi - \phi')) + \underline{u}_{\phi} \frac{\epsilon(z, z') n J_n(k_{np}^{TM} \rho) \sin(n(\phi - \phi'))}{k_{np}^{TM} \rho} \\ & \left. + \underline{u}_z \frac{k_{np}^{TM} J_n(k_{np}^{TM} \rho) \cos(n(\phi - \phi'))}{\gamma_{np}^{TM}} \right\} e^{-\gamma_{np}^{TM} |z-z'|} \\ & + \frac{1}{\pi} \sum_{n=1}^{\infty} \sum_{p=1}^{\infty} \frac{\epsilon(z, z') (k_{np}^{TE})^2 n J_n(k_{np}^{TE} \rho')}{(x_{np}'^2 - n^2) (k_{np}^{TE} \rho') J_n^2(x_{np}')} \left\{ \underline{u}_{\rho} \frac{n J_n(k_{np}^{TE} \rho)}{k_{np}^{TE} \rho} \right. \\ & \cdot \cos(n(\phi - \phi')) + \underline{u}_{\phi} J'_n(k_{np}^{TE} \rho) \sin(n(\phi - \phi')) \left. \right\} e^{-\gamma_{np}^{TE} |z-z'|} \quad (D.19) \end{aligned}$$

Similarly, substitution of (B.22), (B.23), (B.33), (B.34), (B.51), (B.52), (B.62), (B.63), (B.41), (B.59), and (B.54) into (D.18) gives

$$\begin{aligned} H(\underline{Q}, \underline{u}_{\phi} \delta(\underline{r} - \underline{r}')) &= -\frac{jk^2}{2\pi\omega\mu} \sum_{n=0}^{\infty} \sum_{p=1}^{\infty} \frac{\epsilon_n (k_{np}^{TM})^2 J'_n(k_{np}^{TM} \rho')}{\gamma_{np}^{TM} x_{np}^2 J_{n+1}^2(x_{np})} \left\{ \underline{u}_{\rho} \right. \\ & \cdot \frac{n J_n(k_{np}^{TM} \rho)}{k_{np}^{TM} \rho} \sin(n(\phi - \phi')) + \underline{u}_{\phi} J'_n(k_{np}^{TM} \rho) \cos(n(\phi - \phi')) \left. \right\} \\ & \cdot e^{-\gamma_{np}^{TM} |z-z'|} + \frac{j}{\pi\omega\mu} \sum_{n=1}^{\infty} \sum_{p=1}^{\infty} \frac{(k_{np}^{TE})^2 n J_n(k_{np}^{TE} \rho')}{(x_{np}'^2 - n^2) (k_{np}^{TE} \rho') J_n^2(x_{np}')} \left\{ \underline{u}_{\rho} \right. \\ & \cdot \gamma_{np}^{TE} J'_n(k_{np}^{TE} \rho) \sin(n(\phi - \phi')) + \underline{u}_{\phi} \frac{\gamma_{np}^{TE} n J_n(k_{np}^{TE} \rho) \cos(n(\phi - \phi'))}{k_{np}^{TE} \rho} \\ & \left. - \underline{u}_z \epsilon(z, z') k_{np}^{TE} J_n(k_{np}^{TE} \rho) \sin(n(\phi - \phi')) \right\} e^{-\gamma_{np}^{TE} |z-z'|} \quad (D.20) \end{aligned}$$

Consider $\underline{E}(\underline{Q}, \underline{u}_z \delta(\underline{r} - \underline{r}'))$ and $\underline{H}(\underline{Q}, \underline{u}_z \delta(\underline{r} - \underline{r}'))$ evaluated at the point whose cylindrical coordinates are (ρ, ϕ, z) . To obtain expressions for these

fields, we replace the longitudinal magnetic current

$$\underline{M}^v = \underline{u}_z \delta(\underline{r} - \underline{r}') \quad (\text{D.21})$$

by an equivalent transverse electric current [8]. To see that such a "replacement" is possible, recall that $(\underline{E}(\underline{Q}, \underline{M}^v), \underline{H}(\underline{Q}, \underline{M}^v))$ satisfies Maxwell's equations:

$$\left. \begin{aligned} \underline{\nabla} \times \underline{E}(\underline{Q}, \underline{M}^v) &= -j\omega\mu \underline{H}(\underline{Q}, \underline{M}^v) \underline{M}^v \\ \underline{\nabla} \times \underline{H}(\underline{Q}, \underline{M}^v) &= j\omega\epsilon \underline{E}(\underline{Q}, \underline{M}^v) \end{aligned} \right\} \quad (\text{D.22})$$

If

$$\hat{\underline{H}} = \underline{H}(\underline{Q}, \underline{M}^v) + \frac{\underline{M}^v}{j\omega\mu} \quad (\text{D.23})$$

then (D.22) becomes

$$\left. \begin{aligned} \underline{\nabla} \times \underline{E}(\underline{Q}, \underline{M}^v) &= -j\omega\mu \hat{\underline{H}} \\ \underline{\nabla} \times \hat{\underline{H}} &= j\omega\epsilon \underline{E}(\underline{Q}, \underline{M}^v) + \underline{J}^v \end{aligned} \right\} \quad (\text{D.24})$$

where

$$\underline{J}^v = \frac{\underline{\nabla} \times \underline{M}^v}{j\omega\mu} \quad (\text{D.25})$$

If we can find the electromagnetic field $(\underline{E}(\underline{Q}, \underline{M}^v), \hat{\underline{H}})$ that appears in (D.24), then we will, of course, have $\underline{E}(\underline{Q}, \underline{M}^v)$ and, from (D.23), $\underline{H}(\underline{Q}, \underline{M}^v)$ will be given by

$$\underline{H}(\underline{Q}, \underline{M}^v) = \hat{\underline{H}} - \frac{\underline{M}^v}{j\omega\mu} \quad (\text{D.26})$$

From (D.24), the electromagnetic field $(\underline{E}(\underline{Q}, \underline{M}^v), \hat{\underline{H}})$ is the field radiated by the electric current source \underline{J}^v so that we may write

$$\underline{E}(\underline{Q}, \underline{M}^v) = \underline{E}(\underline{J}^v, \underline{Q}) \quad (\text{D.27})$$

$$\hat{\underline{H}} = \underline{H}(\underline{J}^v, \underline{Q}) \quad (\text{D.28})$$

Now, our objective is to find the electromagnetic field $(\underline{E}(\underline{J}^v, \underline{Q}), \underline{H}(\underline{J}^v, \underline{Q}))$.

Unfortunately, \underline{J}^v can not be obtained by substituting (D.21) into (D.25) because $\delta(\underline{r} - \underline{r}')$ is not differentiable. Substitution of (D.5) into (D.21) gives

$$\underline{M}^v = \underline{u}_z \delta(z - z') \delta(\underline{\rho} - \underline{\rho}') \quad (\text{D.29})$$

We attempt to define \underline{M}^v by

$$\underline{M}^v = \underline{u}_z \delta(z - z') \lim_{N \rightarrow \infty} f_N(\underline{\rho} - \underline{\rho}') \quad (\text{D.30})$$

where f_N becomes more and more impulsive as N increases:

$$\lim_{N \rightarrow \infty} f_N(\underline{\rho} - \underline{\rho}') = \delta(\underline{\rho} - \underline{\rho}') \quad (\text{D.31})$$

Now, we can not define $(\underline{E}(\underline{Q}, \underline{M}^v), \underline{H}(\underline{Q}, \underline{M}^v))$ by

$$\left. \begin{aligned} \underline{E}(\underline{Q}, \underline{M}^v) &= \underline{E}(\underline{Q}, \underline{u}_z \delta(z - z') \lim_{N \rightarrow \infty} f_N(\underline{\rho} - \underline{\rho}')) \\ \underline{H}(\underline{Q}, \underline{M}^v) &= \underline{H}(\underline{Q}, \underline{u}_z \delta(z - z') \lim_{N \rightarrow \infty} f_N(\underline{\rho} - \underline{\rho}')) \end{aligned} \right\} \quad (\text{D.32})$$

because, since $\delta(\underline{\rho} - \underline{\rho}')$ is not a legitimate function, the limits indicated in (D.32) do not really exist. The proper definition of $(\underline{E}(\underline{Q}, \underline{M}^v), \underline{H}(\underline{Q}, \underline{M}^v))$ is

$$\left. \begin{aligned} \underline{E}(\underline{Q}, \underline{M}^v) &= \lim_{N \rightarrow \infty} \underline{E}(\underline{Q}, \underline{u}_z \delta(z - z') f_N(\underline{\rho} - \underline{\rho}')) \\ \underline{H}(\underline{Q}, \underline{M}^v) &= \lim_{N \rightarrow \infty} \underline{H}(\underline{Q}, \underline{u}_z \delta(z - z') f_N(\underline{\rho} - \underline{\rho}')) \end{aligned} \right\} \quad (\text{D.33})$$

Taking the approach outlined in the previous paragraph, we replace the fields due to \underline{M}^v of (D.21) by the fields due to

$$\underline{M}^v = \underline{u}_z \delta(z - z') \sum_{n=0}^{N_1} \sum_{p=1}^{N_2} \sum_{s=e,o} C_{np}^{TEs} \psi_{np}^{TEs}(\rho, \phi) \quad (\text{D.34})$$

where the $s = 0$ term is to be omitted when n , the index of the outer summation, is zero. We will choose C_{np}^{TEs} such that \underline{M}^v of (D.34) approaches \underline{M}^v of (D.29) as both N_1 and N_2 increase. Finally, we will take $\lim_{N_1, N_2 \rightarrow \infty}$ of the fields due to \underline{M}^v of (D.34).

In order for \underline{M}^v of (D.34) to approach \underline{M}^v of (D.29), the triple summation in (D.34) must approach $\delta(\underline{\rho} - \underline{\rho}')$. Hence, we write

$$\delta(\underline{\rho} - \underline{\rho}') = \sum_{n=0}^{N_1} \sum_{p=1}^{N_2} \sum_{s=e,o} C_{np}^{TEs} \psi_{np}^{TEs}(\rho, \phi) \quad (\text{D.35})$$

Although it is not strictly true, (D.35) is a means of evaluating C_{np}^{TEs} . Multiplying both sides of (D.35) by $\psi_{mq}^{TEr}(\rho, \phi)$, integrating over the cross section of the waveguide, using the orthogonality relationship [1, secs. 8-1 and 8-2]

$$(k_{np}^{TE})^2 \int_0^a \rho d\rho \int_0^{2\pi} d\phi \psi_{mq}^{TEr}(\rho, \phi) \psi_{np}^{TEs}(\rho, \phi) = \begin{cases} 1, & (r, m, q) = (s, n, p) \\ 0, & \text{otherwise} \end{cases} \quad (\text{D.36})$$

and finally replacing (r, m, q) by (s, n, p) , we obtain

$$C_{np}^{TEs} = (k_{np}^{TE})^2 \psi_{np}^{TEs}(\rho', \phi') \quad (D.37)$$

Substitution of (D.37) into (D.34) gives

$$\underline{M}^v = \underline{u}_z \delta(z - z') \sum_{n=0}^{N_1} \sum_{p=1}^{N_2} \sum_{s=e,o} (k_{np}^{TE})^2 \psi_{np}^{TEs}(\rho', \phi') \psi_{np}^{TEs}(\rho, \phi) \quad (D.38)$$

Substituting (D.38) into (D.25) and using (B.38) and (B.58), we obtain

$$\underline{J}^v = \underline{J} \delta(z - z') \quad (D.39)$$

where

$$\underline{J} = \underline{u}_z \times \frac{1}{j\omega\mu} \sum_{n=0}^{N_1} \sum_{p=1}^{N_2} \sum_{s=e,o} (k_{np}^{TE})^2 \psi_{np}^{TEs}(\rho', \phi') \underline{h}_{np}^{TEs}(\rho, \phi) \quad (D.40)$$

Note that \underline{J} is a surface density of electric current in the $z = z'$ plane.

This paragraph and the next paragraph are devoted to finding the electromagnetic field $(\underline{E}(\underline{Q}, \underline{M}^v), \underline{\hat{H}})$ which satisfies (D.24) with \underline{J}^v given by (D.39). Since the only source is a transverse electric current in the $z = z'$ plane, and since the waveguide is matched at both ends, there will only be waves that travel outward from z' and the transverse component of the electric field will be continuous at z' . Hence, $\underline{E}(\underline{Q}, \underline{M}^v)$ and $\underline{\hat{H}}$ may be expanded as

$$(\underline{E}(\underline{Q}, \underline{M}^v), \underline{\hat{H}}) = \sum_{n=0}^{\infty} \sum_{p=1}^{\infty} \sum_{s=e,o} \left\{ \pm C_{np}^{TM s, J} e^{\pm \gamma_{np}^{TM} z'} \cdot (\underline{E}_{np}^{TM s \pm}, \underline{H}_{np}^{TM s \pm}) + C_{np}^{TE s, J} e^{\pm \gamma_{np}^{TE} z'} (\underline{E}_{np}^{TE s \pm}, \underline{H}_{np}^{TE s \pm}) \right\} \quad (D.41)$$

where the upper sign is to be taken for $z > z'$ and the lower one for $z < z'$. In (D.41), $(\underline{E}_{np}^{TM s \pm}, \underline{H}_{np}^{TM s \pm})$ and $(\underline{E}_{np}^{TE s \pm}, \underline{H}_{np}^{TE s \pm})$ are given by (B.1), (B.2), (B.26), (B.27), (B.35), (B.36), (B.55), and (B.56). Moreover, γ_{np}^{TM} and γ_{np}^{TE} are given by (B.24) and (B.53), respectively. The $s = 0$ term is to be omitted from (D.41) when n , the index of the outer summation, is zero. In (D.41), $C_{np}^{TM s, J}$ and $C_{np}^{TE s, J}$ are unknown constants. The superscript J indicates that

the source is \underline{J} of (D.40). Substituting the expressions in Appendix B for $(\underline{E}^{TM_{s\pm}}, \underline{H}^{TM_{s\pm}})$ and $(\underline{E}^{TE_{s\pm}}, \underline{H}^{TE_{s\pm}})$ into (D.41), we obtain

$$\begin{aligned} \underline{E}(\underline{Q}, \underline{M}^v) = & \sum_{n=0}^{\infty} \sum_{p=1}^{\infty} \sum_{s=e,o} C_{np}^{TM_{s,J}} \left\{ Z_{np}^{TM_{eo}} \right. \\ & \cdot \underline{e}_{np}^{TM_s}(\rho, \phi) + \underline{u}_z \frac{\epsilon(z, z') (k_{np}^{TM})^2 \psi_{np}^{TM_s}(\rho, \phi)}{j\omega\epsilon} \left. \right\} e^{-\gamma_{np}^{TM}|z-z'|} \\ & + \sum_{n=0}^{\infty} \sum_{p=1}^{\infty} \sum_{s=e,o} C_{np}^{TE_{s,J}} \underline{e}_{np}^{TE_s}(\rho, \phi) e^{-\gamma_{np}^{TE}|z-z'|} \end{aligned} \quad (D.42)$$

$$\begin{aligned} \hat{H} = & \sum_{n=0}^{\infty} \sum_{p=1}^{\infty} \sum_{s=e,o} C_{np}^{TM_{s,J}} \epsilon(z, z') \underline{h}_{np}^{TM_s}(\rho, \phi) \\ & \cdot e^{-\gamma_{np}^{TM}|z-z'|} + \sum_{n=0}^{\infty} \sum_{p=1}^{\infty} \sum_{s=e,o} C_{np}^{TE_{s,J}} \left\{ \epsilon(z, z') Y_{np}^{TE_{eo}} \right. \\ & \cdot \underline{h}_{np}^{TE_s}(\rho, \phi) + \underline{u}_z \frac{(k_{np}^{TE})^2 \psi_{np}^{TE_s}(\rho, \phi)}{j\omega\mu} \left. \right\} e^{-\gamma_{np}^{TE}|z-z'|} \end{aligned} \quad (D.43)$$

where $\epsilon(z, z')$ is given by (D.4).

The right-hand side of (D.43) is discontinuous at $z = z'$. The discontinuity is related to \underline{J} of (D.40) by [4, eq. (3-14)]

$$\underline{J} = \underline{u}_z \times (\hat{H}^+ - \hat{H}^-) \quad (D.44)$$

where

$$\hat{H}^+ = \lim_{\substack{z \rightarrow z' \\ z > z'}} \hat{H} \quad (D.45)$$

$$\hat{H}^- = \lim_{\substack{z \rightarrow z' \\ z < z'}} \hat{H} \quad (D.46)$$

Taking $-\underline{u}_z \times$ of both sides of (D.44), we obtain

$$-\underline{u}_z \times \underline{J} = -\underline{u}_z \times \left\{ \underline{u}_z \times (\hat{H}^+ - \hat{H}^-) \right\} \quad (D.47)$$

or, more simply,

$$-\underline{u}_z \times \underline{J} = (\hat{H}^+ - \hat{H}^-)_{\tan} \quad (D.48)$$

where the subscript "tan" denotes the transverse part of the vector. Applying (D.48) to the discontinuous magnetic field of (D.43), we obtain

$$\begin{aligned} -\underline{u}_z \times \underline{J} = & 2 \sum_{n=0}^{\infty} \sum_{p=1}^{\infty} \sum_{s=e,o} C_{np}^{TM_s,J} \underline{h}_{np}^{TM_s}(\rho, \phi) \\ & + 2 \sum_{n=0}^{\infty} \sum_{p=1}^{\infty} \sum_{s=e,o} C_{np}^{TE_s,J} Y_{np}^{TEeo} \underline{h}_{np}^{TE_s}(\rho, \phi) \end{aligned} \quad (D.49)$$

Substitution of (D.40) into (D.49) gives

$$\begin{aligned} & \frac{1}{j\omega\mu} \sum_{n=0}^{\infty} \sum_{p=1}^{N_2} \sum_{s=e,o} (k_{np}^{TE})^2 \psi_{np}^{TE_s}(\rho', \phi') \underline{h}_{np}^{TE_s}(\rho, \phi) = 2 \\ & \cdot \sum_{n=0}^{\infty} \sum_{p=1}^{\infty} \sum_{s=e,o} C_{np}^{TM_s,J} \underline{h}_{np}^{TM_s}(\rho, \phi) + 2 \sum_{n=0}^{\infty} \sum_{p=1}^{\infty} \sum_{s=e,o} C_{np}^{TE_s,J} Y_{np}^{TEeo} \underline{h}_{np}^{TE_s}(\rho, \phi) \end{aligned} \quad (D.50)$$

Because of the orthogonality (B.64) of $\underline{h}_{np}^{TE_s}$ and $\underline{h}_{np}^{TM_s}$, it is evident from (D.50) that, upon using (B.54) to dispose of Y_{np}^{TEeo} ,

$$C_{np}^{TM_s,J} = 0 \quad (D.51)$$

$$C_{np}^{TE_s,J} = \begin{cases} \frac{(k_{np}^{TE})^2 \psi_{np}^{TE_s}(\rho', \phi')}{2\gamma_{np}^{TE}}, & \begin{cases} s = e, o \\ n = 0, 1, \dots, N_1 \\ p = 1, 2, \dots, N_2 \end{cases} \\ 0, & \text{otherwise} \end{cases} \quad (D.52)$$

In view of (B.54), substitution of (D.51) and (D.52) into (D.42) and (D.43) gives

$$\underline{E}(\underline{Q}, \underline{M}^v) = \frac{1}{2} \sum_{n=0}^{N_1} \sum_{p=1}^{N_2} \sum_{s=e,o} \frac{(k_{np}^{TE})^2 \psi_{np}^{TE_s}(\rho', \phi') \underline{h}_{np}^{TE_s}(\rho, \phi) e^{-\gamma_{np}^{TE}|z-z'|}}{\gamma_{np}^{TE}} \quad (D.53)$$

$$\begin{aligned} \hat{H} = & -\frac{j}{2\omega\mu} \sum_{n=0}^{N_1} \sum_{p=1}^{N_2} \sum_{s=e,o} (k_{np}^{TE})^2 \psi_{np}^{TE_s}(\rho', \phi') \\ & \cdot \left\{ \epsilon(z, z') \underline{h}_{np}^{TE_s}(\rho, \phi) + \underline{u}_z \frac{(k_{np}^{TE})^2 \psi_{np}^{TE_s}(\rho, \phi)}{\gamma_{np}^{TE}} \right\} e^{-\gamma_{np}^{TE}|z-z'|} \end{aligned} \quad (D.54)$$

Substituting (B.41), (B.59), (B.51), and (B.62) into (D.53), taking $\lim_{N_1, N_2 \rightarrow \infty}$, and then substituting (D.21) for \underline{M}^v , we obtain

$$\underline{E}(\underline{Q}, \underline{u}_z \delta(\underline{r} - \underline{r}')) = \frac{1}{2\pi} \sum_{n=0}^{\infty} \sum_{p=1}^{\infty} \frac{\epsilon_n (k_{np}^{TE})^3 J_n(k_{np}^{TE} \rho')}{\gamma_{np}^{TE} (x_{np}'^2 - n^2) J_n^2(x_{np}') } \left\{ \underline{u}_\rho \frac{n J_n(k_{np}^{TE} \rho)}{k_{np}^{TE} \rho} \right.$$

$$\cdot \sin(n(\phi - \phi')) + \underline{u}_\phi J'_n(k_{np}^{TE} \rho) \cos(n(\phi - \phi')) \} e^{-\gamma_{np}^{TE}|z-z'|} \quad (D.55)$$

Substituting (B.41), (B.59), (B.52), and (B.63) into (D.54), we obtain

$$\begin{aligned} \hat{H} = & \frac{j}{2\pi\omega\mu} \sum_{n=0}^{N_1} \sum_{p=1}^{N_2} \frac{\epsilon_n(k_{np}^{TE})^3 J_n(k_{np}^{TE} \rho')}{(x_{np}'^2 - n^2) J_n^2(x_{np}') } \left\{ \epsilon(z, z') \right. \\ & \cdot \left(\underline{u}_\rho J'_n(k_{np}^{TE} \rho) \cos(n(\phi - \phi')) - \underline{u}_\phi \frac{n J_n(k_{np}^{TE} \rho) \sin(n(\phi - \phi'))}{k_{np}^{TE} \rho} \right) \\ & \left. - \underline{u}_z \frac{k_{np}^{TE} J_n(k_{np}^{TE} \rho) \cos(n(\phi - \phi'))}{\gamma_{np}^{TE}} \right\} e^{-\gamma_{np}^{TE}|z-z'|} \quad (D.56) \end{aligned}$$

We want to substitute \hat{H} of (D.56) and \underline{M}^v of (D.38) into (D.26) before letting N_1 and N_2 approach ∞ . Substitution of (B.41) and (B.59) into (D.38) gives

$$\underline{M}^v = \underline{u}_z \frac{\delta(z - z')}{\pi} \sum_{n=0}^{N_1} \sum_{p=1}^{N_2} \frac{\epsilon_n(k_{np}^{TE})^2 J_n(k_{np}^{TE} \rho') J_n(k_{np}^{TE} \rho) \cos(n(\phi - \phi'))}{(x_{np}'^2 - n^2) J_n^2(x_{np}') } \quad (D.57)$$

Substituting (D.56) and (D.57) into the right-hand side of (D.26), taking $\lim_{N_1, N_2 \rightarrow \infty}$, and then substituting (D.21) into the left-hand side of (D.26), we obtain

$$\begin{aligned} H(0, \underline{u}_z \delta(r - r')) = & \frac{j}{2\pi\omega\mu} \sum_{n=0}^{\infty} \sum_{p=1}^{\infty} \frac{\epsilon_n(k_{np}^{TE})^3 J_n(k_{np}^{TE} \rho')}{(x_{np}'^2 - n^2) J_n^2(x_{np}') } \left\{ \left\{ \epsilon(z, z') \right. \right. \\ & \cdot \left(\underline{u}_\rho J'_n(k_{np}^{TE} \rho) \cos(n(\phi - \phi')) - \underline{u}_\phi \frac{n J_n(k_{np}^{TE} \rho) \sin(n(\phi - \phi'))}{k_{np}^{TE} \rho} \right) \\ & \left. \left. - \underline{u}_z \frac{k_{np}^{TE} J_n(k_{np}^{TE} \rho) \cos(n(\phi - \phi'))}{\gamma_{np}^{TE}} \right\} e^{-\gamma_{np}^{TE}|z-z'|} \right. \\ & \left. + \underline{u}_z \frac{2\delta(z - z') J_n(k_{np}^{TE} \rho) \cos(n(\phi - \phi'))}{k_{np}^{TE}} \right\} \quad (D.58) \end{aligned}$$

There are two \underline{u}_z terms inside the double summation on the right-hand side of (D.58). The second \underline{u}_z term came from the $-\underline{M}^v/(j\omega\mu)$ term on the right-hand side of (D.26). Therefore, if we deleted the second \underline{u}_z term from the right-hand side of (D.58), we would have \hat{H} instead of $H(0, \underline{u}_z \delta(r - r'))$. Equation (8) in [9, sec. 22] correctly gives $\underline{E}(0, \underline{u}_z \delta(r - r'))$ of (D.55) for the

electric field of the magnetic current element $\mathbf{u}_z \delta(\mathbf{r} - \mathbf{r}')$. However, (8) in [9, sec. 22] incorrectly gives the previously mentioned \hat{H} for the magnetic field of the magnetic current element $\mathbf{u}_z \delta(\mathbf{r} - \mathbf{r}')$. This error is pointed out in [10] and [11].

In Appendix D, we found that $\underline{E}(\mathbf{Q}, \mathbf{u}_\phi \delta(\mathbf{r} - \mathbf{r}'))$, $\underline{H}(\mathbf{Q}, \mathbf{u}_\phi \delta(\mathbf{r} - \mathbf{r}'))$, $\underline{E}(\mathbf{Q}, \mathbf{u}_z \delta(\mathbf{r} - \mathbf{r}'))$, and $\underline{H}(\mathbf{Q}, \mathbf{u}_z \delta(\mathbf{r} - \mathbf{r}'))$ are given by (D.19), (D.20), (D.55), and (D.58), respectively. In these equations, k_{np}^{TM} , γ_{np}^{TM} , k_{np}^{TE} , and γ_{np}^{TE} are given by (B.7), (B.24), (B.41), and (B.53), respectively.

Appendix E

Evaluation of Integrals with Respect to ϕ' and ϕ

In Appendix E, the integrals (4.57)–(4.60) and (4.100)–(4.103) are evaluated. The integrals (4.57)–(4.60) are

$$\phi^{\gamma 1} = \int_{\phi_1}^{\phi_2} \sin\left(\frac{p\pi y'^{\gamma+}}{b}\right) \sin(r(\phi - \phi')) d\phi' \quad (\text{E.1})$$

$$\phi^{\gamma 2} = \int_{\phi_1}^{\phi_2} \sin\left(\frac{p\pi y'^{\gamma+}}{b}\right) \cos(r(\phi - \phi')) d\phi' \quad (\text{E.2})$$

$$\phi^{\gamma 3} = \int_{\phi_1}^{\phi_2} \cos\left(\frac{p\pi y'^{\gamma+}}{b}\right) \sin(r(\phi - \phi')) d\phi' \quad (\text{E.3})$$

$$\phi^{\gamma 4} = \int_{\phi_1}^{\phi_2} \cos\left(\frac{p\pi y'^{\gamma+}}{b}\right) \cos(r(\phi - \phi')) d\phi' \quad (\text{E.4})$$

In the above equations, γ is 1 or 2, y'^{1+} is given by (4.39), y'^{2+} by (4.40), ϕ_1 by (4.42), and ϕ_2 by (4.43).

Changing the variable of integration from ϕ' to $y'^{\gamma+}$, (E.1)–(E.4) become

$$\phi^{\gamma 1} = -\frac{(-1)^\gamma}{x_o} \int_0^b \sin\left(\frac{p\pi y'^{\gamma+}}{b}\right) \sin\left(\frac{r}{x_o}(y'^{\gamma+} - y^{\gamma+})\right) dy'^{\gamma+} \quad (\text{E.5})$$

$$\phi^{\gamma 2} = \frac{1}{x_o} \int_0^b \sin\left(\frac{p\pi y'^{\gamma+}}{b}\right) \cos\left(\frac{r}{x_o}(y'^{\gamma+} - y^{\gamma+})\right) dy'^{\gamma+} \quad (\text{E.6})$$

$$\phi^{\gamma 3} = -\frac{(-1)^\gamma}{x_o} \int_0^b \cos\left(\frac{p\pi y'^{\gamma+}}{b}\right) \sin\left(\frac{r}{x_o}(y'^{\gamma+} - y^{\gamma+})\right) dy'^{\gamma+} \quad (\text{E.7})$$

$$\phi^{\gamma 4} = \frac{1}{x_o} \int_0^b \cos\left(\frac{p\pi y^{\gamma+}}{b}\right) \cos\left(\frac{r}{x_o}(y^{\gamma+} - y^{\gamma+})\right) dy^{\gamma+} \quad (\text{E.8})$$

where $y^{\gamma+}$ is given by (2.15) and (2.16). From (1.2) and (2.18), x_o is given by

$$x_o = \frac{b}{2\phi_o} \quad (\text{E.9})$$

Using [5, formulas 401.02 and 401.04] to expand the trigonometric functions of the difference arguments in (E.5)–(E.8), we obtain

$$\phi^{\gamma 1} = -(-1)^\gamma \left\{ \phi_p^{(1)} \cos\left(\frac{ry^{\gamma+}}{x_o}\right) - \phi_p^{(2)} \sin\left(\frac{ry^{\gamma+}}{x_o}\right) \right\} \quad (\text{E.10})$$

$$\phi^{\gamma 2} = \phi_p^{(2)} \cos\left(\frac{ry^{\gamma+}}{x_o}\right) + \phi_p^{(1)} \sin\left(\frac{ry^{\gamma+}}{x_o}\right) \quad (\text{E.11})$$

$$\phi^{\gamma 3} = -(-1)^\gamma \left\{ \phi_p^{(3)} \cos\left(\frac{ry^{\gamma+}}{x_o}\right) - \phi_p^{(4)} \sin\left(\frac{ry^{\gamma+}}{x_o}\right) \right\} \quad (\text{E.12})$$

$$\phi^{\gamma 4} = \phi_p^{(4)} \cos\left(\frac{ry^{\gamma+}}{x_o}\right) + \phi_p^{(3)} \sin\left(\frac{ry^{\gamma+}}{x_o}\right) \quad (\text{E.13})$$

where

$$\phi_p^{(1)} = \frac{1}{x_o} \int_0^b \sin\left(\frac{p\pi x}{b}\right) \sin\left(\frac{rx}{x_o}\right) dx \quad (\text{E.14})$$

$$\phi_p^{(2)} = \frac{1}{x_o} \int_0^b \sin\left(\frac{p\pi x}{b}\right) \cos\left(\frac{rx}{x_o}\right) dx \quad (\text{E.15})$$

$$\phi_p^{(3)} = \frac{1}{x_o} \int_0^b \cos\left(\frac{p\pi x}{b}\right) \sin\left(\frac{rx}{x_o}\right) dx \quad (\text{E.16})$$

$$\phi_p^{(4)} = \frac{1}{x_o} \int_0^b \cos\left(\frac{p\pi x}{b}\right) \cos\left(\frac{rx}{x_o}\right) dx \quad (\text{E.17})$$

The integrals in (E.14)–(E.17) are evaluated by using the integration formulas [5, formulas 435, 445, and 465]

$$\int \sin(ax) \sin(bx) dx = \frac{1}{2} \left\{ \frac{\sin((a-b)x)}{a-b} - \frac{\sin((a+b)x)}{a+b} \right\} \quad (\text{E.18})$$

$$\int \cos(ax) \cos(bx) dx = \frac{1}{2} \left\{ \frac{\sin((a-b)x)}{a-b} + \frac{\sin((a+b)x)}{a+b} \right\} \quad (\text{E.19})$$

$$\int \sin(ax) \cos(bx) dx = -\frac{1}{2} \left\{ \frac{\cos((a-b)x)}{a-b} + \frac{\cos((a+b)x)}{a+b} \right\} \quad (\text{E.20})$$

If $|a| = |b|$, then the right-hand sides of (E.18)–(E.20) are to be replaced by their limits as $|b|$ approaches $|a|$. Unfortunately, (E.20) is subject to excessive roundoff error when either $(a - b)$ or $(a + b)$ is small. To avoid this error, we use the trigonometric identity [5, formula 403.22]

$$\cos x = 1 - 2 \sin^2\left(\frac{x}{2}\right) \quad (\text{E.21})$$

to recast (E.20) as

$$\int \sin(ax) \cos(bx) dx = \frac{\sin^2\left\{\frac{(a-b)x}{2}\right\}}{a-b} + \frac{\sin^2\left\{\frac{(a+b)x}{2}\right\}}{a+b} \quad (\text{E.22})$$

The “ b ” which appears in (E.18)–(E.20) and (E.22) is not to be confused with that in (E.14)–(E.17).

Using (E.18), (E.19), and (E.22) to evaluate the integrals in (E.14)–(E.17), we obtain

$$\phi_p^{(1)} = \frac{b}{2x_o} \left\{ \frac{\sin\left(p\pi - \frac{rb}{x_o}\right)}{p\pi - \frac{rb}{x_o}} - \frac{\sin\left(p\pi + \frac{rb}{x_o}\right)}{p\pi + \frac{rb}{x_o}} \right\} \quad (\text{E.23})$$

$$\phi_p^{(2)} = \frac{b}{x_o} \left\{ \frac{\sin^2\left(\frac{1}{2}\left(p\pi - \frac{rb}{x_o}\right)\right)}{p\pi - \frac{rb}{x_o}} + \frac{\sin^2\left(\frac{1}{2}\left(p\pi + \frac{rb}{x_o}\right)\right)}{p\pi + \frac{rb}{x_o}} \right\} \quad (\text{E.24})$$

$$\phi_p^{(3)} = \frac{b}{x_o} \left\{ -\frac{\sin^2\left(\frac{1}{2}\left(p\pi - \frac{rb}{x_o}\right)\right)}{p\pi - \frac{rb}{x_o}} + \frac{\sin^2\left(\frac{1}{2}\left(p\pi + \frac{rb}{x_o}\right)\right)}{p\pi + \frac{rb}{x_o}} \right\} \quad (\text{E.25})$$

$$\phi_p^{(4)} = \frac{b}{2x_o} \left\{ \frac{\sin\left(p\pi - \frac{rb}{x_o}\right)}{p\pi - \frac{rb}{x_o}} + \frac{\sin\left(p\pi + \frac{rb}{x_o}\right)}{p\pi + \frac{rb}{x_o}} \right\} \quad (\text{E.26})$$

If $(p\pi \pm \frac{rb}{x_o})$ is zero, then the right-hand sides of (E.23)–(E.26) are to be replaced by their limits as $(p\pi \pm \frac{rb}{x_o})$ approaches zero.

The integrals (4.100)–(4.103) are

$$\phi^{\alpha\gamma 1} = \int_{\phi_3}^{\phi_4} \phi^{\gamma 1} \cos\left(\frac{m\pi y^{\alpha+}}{b}\right) d\phi \quad (\text{E.27})$$

$$\phi^{\alpha\gamma 2} = \int_{\phi_3}^{\phi_4} \phi^{\gamma 2} \sin\left(\frac{m\pi y^{\alpha+}}{b}\right) d\phi \quad (\text{E.28})$$

$$\phi^{\alpha\gamma 3} = \int_{\phi_3}^{\phi_4} \phi^{\gamma 3} \sin\left(\frac{m\pi y^{\alpha+}}{b}\right) d\phi \quad (\text{E.29})$$

$$\phi^{\alpha\gamma 4} = \int_{\phi_3}^{\phi_4} \phi^{\gamma 4} \cos\left(\frac{m\pi y^{\alpha+}}{b}\right) d\phi \quad (\text{E.30})$$

where γ is 1 or 2, α is 1 or 2, y^{1+} is given by (2.15), y^{2+} by (2.16), ϕ_3 by (4.104), and ϕ_4 by (4.105). In (2.15) and (2.16), x_o is given by (2.18) where ϕ_o is given by (1.2). In (E.27)–(E.30), $\phi^{\gamma 1}$, $\phi^{\gamma 2}$, $\phi^{\gamma 3}$, and $\phi^{\gamma 4}$ are given by (E.10)–(E.13), respectively. Substitution of (E.10)–(E.13) into (E.27)–(E.30) gives

$$\phi^{\alpha\gamma 1} = -(-1)^\gamma \left\{ \phi_p^{(1)} \phi^{\alpha 2\gamma 2} - \phi_p^{(2)} \phi^{\alpha 2\gamma 1} \right\} \quad (\text{E.31})$$

$$\phi^{\alpha\gamma 2} = \phi_p^{(2)} \phi^{\alpha 1\gamma 2} + \phi_p^{(1)} \phi^{\alpha 1\gamma 1} \quad (\text{E.32})$$

$$\phi^{\alpha\gamma 3} = -(-1)^\gamma \left\{ \phi_p^{(3)} \phi^{\alpha 1\gamma 2} - \phi_p^{(4)} \phi^{\alpha 1\gamma 1} \right\} \quad (\text{E.33})$$

$$\phi^{\alpha\gamma 4} = \phi_p^{(4)} \phi^{\alpha 2\gamma 2} + \phi_p^{(3)} \phi^{\alpha 2\gamma 1} \quad (\text{E.34})$$

where

$$\phi^{\alpha 1\gamma 1} = \int_{\phi_3}^{\phi_4} \sin\left(\frac{m\pi y^{\alpha+}}{b}\right) \sin\left(\frac{r y^{\gamma+}}{x_o}\right) d\phi \quad (\text{E.35})$$

$$\phi^{\alpha 2\gamma 1} = \int_{\phi_3}^{\phi_4} \cos\left(\frac{m\pi y^{\alpha+}}{b}\right) \sin\left(\frac{r y^{\gamma+}}{x_o}\right) d\phi \quad (\text{E.36})$$

$$\phi^{\alpha 1\gamma 2} = \int_{\phi_3}^{\phi_4} \sin\left(\frac{m\pi y^{\alpha+}}{b}\right) \cos\left(\frac{r y^{\gamma+}}{x_o}\right) d\phi \quad (\text{E.37})$$

$$\phi^{\alpha 2\gamma 2} = \int_{\phi_3}^{\phi_4} \cos\left(\frac{m\pi y^{\alpha+}}{b}\right) \cos\left(\frac{r y^{\gamma+}}{x_o}\right) d\phi \quad (\text{E.38})$$

Seeking to change the variable of integration in (E.35)–(E.38) from ϕ to $y^{\alpha+}$, we differentiate (2.15) and (2.16) to obtain

$$d\phi = \begin{cases} -\frac{dy^{\alpha+}}{x_o}, & \alpha = 1 \\ \frac{dy^{\alpha+}}{x_o}, & \alpha = 2 \end{cases} \quad (\text{E.39})$$

Substituting (4.104) and (4.105) into (2.15) and (2.16) and using (E.9), we obtain

$$y^{\alpha+} \Big|_{\phi=\phi_3} = \begin{cases} b, & \alpha = 1 \\ 0, & \alpha = 2 \end{cases} \quad (\text{E.40})$$

$$y^{\alpha+}]_{\phi=\phi_4} = \begin{cases} 0, & \alpha = 1 \\ b, & \alpha = 2 \end{cases} \quad (\text{E.41})$$

Applying (E.39)–(E.41) to (E.35)–(E.38), we find that

$$\phi^{\alpha 1 \gamma 1} = \frac{1}{x_o} \int_0^b \sin\left(\frac{m\pi y^{\alpha+}}{b}\right) \sin\left(\frac{ry^{\gamma+}}{x_o}\right) dy^{\alpha+} \quad (\text{E.42})$$

$$\phi^{\alpha 2 \gamma 1} = \frac{1}{x_o} \int_0^b \cos\left(\frac{m\pi y^{\alpha+}}{b}\right) \sin\left(\frac{ry^{\gamma+}}{x_o}\right) dy^{\alpha+} \quad (\text{E.43})$$

$$\phi^{\alpha 1 \gamma 2} = \frac{1}{x_o} \int_0^b \sin\left(\frac{m\pi y^{\alpha+}}{b}\right) \cos\left(\frac{ry^{\gamma+}}{x_o}\right) dy^{\alpha+} \quad (\text{E.44})$$

$$\phi^{\alpha 2 \gamma 2} = \frac{1}{x_o} \int_0^b \cos\left(\frac{m\pi y^{\alpha+}}{b}\right) \cos\left(\frac{ry^{\gamma+}}{x_o}\right) dy^{\alpha+} \quad (\text{E.45})$$

if $\gamma = \alpha$, then (E.42)–(E.45) can be expressed as

$$\phi^{\alpha 1 \gamma 1} = \phi_m^{(1)}, \quad \gamma = \alpha \quad (\text{E.46})$$

$$\phi^{\alpha 2 \gamma 1} = \phi_m^{(3)}, \quad \gamma = \alpha \quad (\text{E.47})$$

$$\phi^{\alpha 1 \gamma 2} = \phi_m^{(2)}, \quad \gamma = \alpha \quad (\text{E.48})$$

$$\phi^{\alpha 2 \gamma 2} = \phi_m^{(4)}, \quad \gamma = \alpha \quad (\text{E.49})$$

where the ϕ 's on the right-hand sides of (E.46)–(E.49) are given by (E.14)–(E.17) with p replaced by m . If $\gamma \neq \alpha$, we add (2.16) to (2.15) in order to obtain

$$y^{\gamma+} = \pi x_o + b - y^{\alpha+}, \quad \gamma \neq \alpha \quad (\text{E.50})$$

Substituting (E.50) into (E.42)–(E.45) and using

$$\begin{aligned} \sin\left(r\pi + \frac{r}{x_o}(b - y^{\alpha+})\right) &= (-1)^r \\ &\cdot \left\{ \sin\left(\frac{rb}{x_o}\right) \cos\left(\frac{ry^{\alpha+}}{x_o}\right) - \cos\left(\frac{rb}{x_o}\right) \sin\left(\frac{ry^{\alpha+}}{x_o}\right) \right\} \end{aligned} \quad (\text{E.51})$$

$$\begin{aligned} \cos\left(r\pi + \frac{r}{x_o}(b - y^{\alpha+})\right) &= (-1)^r \\ &\cdot \left\{ \cos\left(\frac{rb}{x_o}\right) \cos\left(\frac{ry^{\alpha+}}{x_o}\right) + \sin\left(\frac{rb}{x_o}\right) \sin\left(\frac{ry^{\alpha+}}{x_o}\right) \right\} \end{aligned} \quad (\text{E.52})$$

we obtain

$$\phi^{\alpha 1 \gamma 1} = (-1)^r \left\{ \phi_m^{(2)} \sin\left(\frac{rb}{x_o}\right) - \phi_m^{(1)} \cos\left(\frac{rb}{x_o}\right) \right\}, \quad \gamma \neq \alpha \quad (\text{E.53})$$

$$\phi^{\alpha 2 \gamma 1} = (-1)^r \left\{ \phi_m^{(4)} \sin\left(\frac{rb}{x_o}\right) - \phi_m^{(3)} \cos\left(\frac{rb}{x_o}\right) \right\}, \quad \gamma \neq \alpha \quad (\text{E.54})$$

$$\phi^{\alpha 1 \gamma 2} = (-1)^r \left\{ \phi_m^{(2)} \cos\left(\frac{rb}{x_o}\right) + \phi_m^{(1)} \sin\left(\frac{rb}{x_o}\right) \right\}, \quad \gamma \neq \alpha \quad (\text{E.55})$$

$$\phi^{\alpha 2 \gamma 2} = (-1)^r \left\{ \phi_m^{(4)} \cos\left(\frac{rb}{x_o}\right) + \phi_m^{(3)} \sin\left(\frac{rb}{x_o}\right) \right\}, \quad \gamma \neq \alpha \quad (\text{E.56})$$

where the ϕ 's on the right-hand sides of (E.53)–(E.56) are given by (E.14)–(E.17) with p replaced by m . These ϕ 's can be calculated from (E.23)–(E.26).

The results obtained in Appendix E are stated as follows. The quantities $\phi^{\gamma 1}$, $\phi^{\gamma 2}$, $\phi^{\gamma 3}$, and $\phi^{\gamma 4}$ are given by (E.10)–(E.13) in which $\phi_p^{(1)}$, $\phi_p^{(2)}$, $\phi_p^{(3)}$, and $\phi_p^{(4)}$ are given by (E.23)–(E.26). The quantities $\phi^{\alpha \gamma 1}$, $\phi^{\alpha \gamma 2}$, $\phi^{\alpha \gamma 3}$, and $\phi^{\alpha \gamma 4}$ are given by (E.31)–(E.34) in which $\phi_p^{(1)}$, $\phi_p^{(2)}$, $\phi_p^{(3)}$, and $\phi_p^{(4)}$ are given by (E.23)–(E.26). Moreover, $\phi^{\alpha 1 \gamma 1}$, $\phi^{\alpha 2 \gamma 1}$, $\phi^{\alpha 1 \gamma 2}$, and $\phi^{\alpha 2 \gamma 2}$ are given by (E.46)–(E.48) for $\gamma = \alpha$ and by (E.53)–(E.56) for $\gamma \neq \alpha$.

Appendix F

Evaluation of Integrals with Respect to z' and z

In Appendix F, the integrals (4.77)–(4.88) are evaluated for $(-\frac{\epsilon}{2} \leq z \leq \frac{\epsilon}{2})$. Afterwards, the integrals (4.106)–(4.110) are evaluated. For $(-\frac{\epsilon}{2} \leq z \leq \frac{\epsilon}{2})$, the integrals (4.77)–(4.88) are

$$z^{\delta 1} = \int_{-\frac{\epsilon}{2}}^z e^{-\gamma_{rs}^{\delta}(L_3 - z')} \cos\left(\frac{q\pi z'^+}{c}\right) dz' \quad (F.1)$$

$$z^{\delta 2} = \int_z^{\frac{\epsilon}{2}} \cosh\left(\gamma_{rs}^{\delta}(L_3 - z')\right) \cos\left(\frac{q\pi z'^+}{c}\right) dz' \quad (F.2)$$

$$z^{TE3} = \int_{-\frac{\epsilon}{2}}^z e^{-\gamma_{rs}^{TE}(L_3 - z')} \sin\left(\frac{q\pi z'^+}{c}\right) dz' \quad (F.3)$$

$$z^{TE4} = \int_z^{\frac{\epsilon}{2}} \sinh\left(\gamma_{rs}^{TE}(L_3 - z')\right) \sin\left(\frac{q\pi z'^+}{c}\right) dz' \quad (F.4)$$

where z'^+ is given by (4.41). In (F.1) and (F.2), δ is either TM or TE .

Substituting

$$\gamma_{rs}^{\delta} = j\beta_{rs}^{\delta}, \quad \delta = TM, TE \quad (F.5)$$

in (F.1)–(F.4), using [5, formulas 654.6 and 654.7], changing the variable of integration from z' to z'^+ , and finally replacing z'^+ by x , we obtain

$$z^{\delta 1} = \int_0^{z^+} \left\{ \cos\left(\beta_{rs}^{\delta}(L_3^+ - x)\right) - j \sin\left(\beta_{rs}^{\delta}(L_3^+ - x)\right) \right\} \cos\left(\frac{q\pi x}{c}\right) dx \quad (F.6)$$

$$z^{\delta 2} = \int_{z^+}^c \cos\left(\beta_{rs}^{\delta}(L_3^+ - x)\right) \cos\left(\frac{q\pi x}{c}\right) dx \quad (F.7)$$

$$z^{TE3} = \int_0^{z^+} \left\{ \cos(\beta_{rs}^{TE}(L_3^+ - x)) - j \sin(\beta_{rs}^{TE}(L_3^+ - x)) \right\} \sin\left(\frac{q\pi x}{c}\right) dx \quad (F.8)$$

$$z^{TE4} = j \int_{z^+}^c \sin(\beta_{rs}^{TE}(L_3^+ - x)) \sin\left(\frac{q\pi x}{c}\right) dx \quad (F.9)$$

where

$$z^+ = z + \frac{c}{2} \quad (F.10)$$

$$L_3^+ = L_3 + \frac{c}{2} \quad (F.11)$$

The sum of (F.6) and (F.7) is

$$\begin{aligned} z^{\delta 1} + z^{\delta 2} &= \int_0^c \cos(\beta_{rs}^{\delta}(L_3^+ - x)) \cos\left(\frac{q\pi x}{c}\right) dx \\ &\quad - j \int_0^{z^+} \sin(\beta_{rs}^{\delta}(L_3^+ - x)) \cos\left(\frac{q\pi x}{c}\right) dx \end{aligned} \quad (F.12)$$

The difference between (F.8) and (F.9) is

$$\begin{aligned} z^{TE3} - z^{TE4} &= \int_0^{z^+} \cos(\beta_{rs}^{TE}(L_3^+ - x)) \sin\left(\frac{q\pi x}{c}\right) dx \\ &\quad - j \int_0^c \sin(\beta_{rs}^{TE}(L_3^+ - x)) \sin\left(\frac{q\pi x}{c}\right) dx \end{aligned} \quad (F.13)$$

Since the right-hand side of (F.12) is simpler than that of (F.6), we will obtain $z^{\delta 1}$ by evaluating $(z^{\delta 1} + z^{\delta 2})$ and $z^{\delta 2}$ and by setting

$$z^{\delta 1} = (z^{\delta 1} + z^{\delta 2}) - z^{\delta 2} \quad (F.14)$$

Similarly, we will obtain z^{TE3} by evaluating $(z^{TE3} - z^{TE4})$ and z^{TE4} and by setting

$$z^{TE3} = (z^{TE3} - z^{TE4}) + z^{TE4} \quad (F.15)$$

The integrals in (F.7), (F.12), (F.9), and (F.13) are of the forms [12, formulas 2.532]

$$\begin{aligned} \int \sin(ax + b) \sin(cx + d) dx &= \frac{\sin((a - c)x + b - d)}{2(a - c)} \\ &\quad - \frac{\sin((a + c)x + b + d)}{2(a + c)} \end{aligned} \quad (F.16)$$

$$\int \sin(ax + b) \cos(cx + d) dx = -\frac{\cos((a - c)x + b - d)}{2(a - c)} - \frac{\cos((a + c)x + b + d)}{2(a + c)} \quad (\text{F.17})$$

$$\int \cos(ax + b) \cos(cx + d) dx = \frac{\sin((a - c)x + b - d)}{2(a - c)} + \frac{\sin((a + c)x + b + d)}{2(a + c)} \quad (\text{F.18})$$

In (F.16)–(F.18), a , b , and c are arbitrary constants not to be confused with the specific dimensions a , b , and c in Fig. 1.1. If $a = \pm c$, then the right-hand sides of (F.16)–(F.18) are to be replaced by their limits as a approaches $\pm c$.

Using (F.16)–(F.18) to evaluate the integrals in (F.7), (F.12), (F.9), and (F.13), we obtain

$$z^{\delta 2} = \frac{\sin(q^{\delta -}c + \beta_{rs}^{\delta}L_3^+) - \sin(q^{\delta -}z^+ + \beta_{rs}^{\delta}L_3^+)}{2q^{\delta -}} + \frac{\sin(q^{\delta +}c - \beta_{rs}^{\delta}L_3^+) - \sin(q^{\delta +}z^+ - \beta_{rs}^{\delta}L_3^+)}{2q^{\delta +}} \quad (\text{F.19})$$

$$z^{\delta 1} + z^{\delta 2} = \frac{\sin(q^{\delta -}c + \beta_{rs}^{\delta}L_3^+) - je^{-j\beta_{rs}^{\delta}L_3^+} + j\cos(q^{\delta -}z^+ + \beta_{rs}^{\delta}L_3^+)}{2q^{\delta -}} + \frac{\sin(q^{\delta +}c - \beta_{rs}^{\delta}L_3^+) + je^{-j\beta_{rs}^{\delta}L_3^+} - j\cos(q^{\delta +}z^+ - \beta_{rs}^{\delta}L_3^+)}{2q^{\delta +}} \quad (\text{F.20})$$

$$z^{TE4} = -j\frac{\sin(q^{TE-}c + \beta_{rs}^{TE}L_3^+) - \sin(q^{TE-}z^+ + \beta_{rs}^{TE}L_3^+)}{2q^{TE-}} + j\frac{\sin(q^{TE+}c - \beta_{rs}^{TE}L_3^+) - \sin(q^{TE+}z^+ - \beta_{rs}^{TE}L_3^+)}{2q^{TE+}} \quad (\text{F.21})$$

$$z^{TE3} - z^{TE4} = \frac{j\sin(q^{TE-}c + \beta_{rs}^{TE}L_3^+) + e^{-j\beta_{rs}^{TE}L_3^+} - \cos(q^{TE-}z^+ + \beta_{rs}^{TE}L_3^+)}{2q^{TE-}} + \frac{-j\sin(q^{TE+}c - \beta_{rs}^{TE}L_3^+) + e^{-j\beta_{rs}^{TE}L_3^+} - \cos(q^{TE+}z^+ - \beta_{rs}^{TE}L_3^+)}{2q^{TE+}} \quad (\text{F.22})$$

where

$$q^{\delta -} = \frac{q\pi}{c} - \beta_{rs}^{\delta}, \quad \delta = TM, TE \quad (\text{F.23})$$

$$q^{\delta+} = \frac{q\pi}{c} + \beta_{rs}^{\delta}, \quad \delta = TM, TE \quad (F.24)$$

Substituting (F.19) and (F.20) into (F.14) and using (E.21), we arrive at

$$z^{\delta 1} = \frac{e^{-j\beta_{rs}^{\delta} L_3^+}}{2} \left\{ \frac{\sin(q^{\delta-} z^+) - 2j \sin^2(\frac{q^{\delta-} z^+}{2})}{q^{\delta-}} + \frac{\sin(q^{\delta+} z^+) + 2j \sin^2(\frac{q^{\delta+} z^+}{2})}{q^{\delta+}} \right\} \quad (F.25)$$

Using [5, formulas 401.01 and 401.02] to expand the trigonometric functions in (F.19) and applying (E.21), we obtain

$$z^{\delta 2} = \left[\left\{ \left\{ \sin(q^{\delta-} c) - \sin(q^{\delta-} z^+) \right\} \cos(\beta_{rs}^{\delta} L_3^+) - 2 \left\{ \sin^2(\frac{q^{\delta-} c}{2}) - \sin^2(\frac{q^{\delta-} z^+}{2}) \right\} \sin(\beta_{rs}^{\delta} L_3^+) \right\} / (2q^{\delta-}) \right] + \left[\left\{ \left\{ \sin(q^{\delta+} c) - \sin(q^{\delta+} z^+) \right\} \cos(\beta_{rs}^{\delta} L_3^+) + 2 \left\{ \sin^2(\frac{q^{\delta+} c}{2}) - \sin^2(\frac{q^{\delta+} z^+}{2}) \right\} \sin(\beta_{rs}^{\delta} L_3^+) \right\} / (2q^{\delta+}) \right] \quad (F.26)$$

If $q^{\delta\pm} = 0$, then the right-hand sides of (F.25) and (F.26) are to be replaced by their limits as $q^{\delta\pm}$ approaches zero. If β_{rs}^{δ} is purely real, expressions (F.25) and (F.26) are suitable for calculating (4.74), (4.75), (4.89), and (4.90) at particular values of z because (F.25) and (F.26) are not subject to excessive roundoff error when $|q^{\delta\pm}|$ is small.

If β_{rs}^{δ} is purely imaginary, we use (F.5) to obtain

$$\beta_{rs}^{\delta} = -j\gamma_{rs}^{\delta} \quad (F.27)$$

Substituting (F.27) into (F.25) and using (E.21) to dispose of the \sin^2 terms, we arrive at

$$z^{\delta 1} = \frac{je^{-\gamma_{rs}^{\delta} L_3^+}}{2} \left\{ \frac{e^{-jq^{\delta-} z^+} - 1}{q^{\delta-}} + \frac{1 - e^{jq^{\delta+} z^+}}{q^{\delta+}} \right\} \quad (F.28)$$

where

$$q^{\delta-} = \frac{q\pi}{c} + j\gamma_{rs}^{\delta} \quad (F.29)$$

$$q^{\delta+} = \frac{q\pi}{c} - j\gamma_{rs}^{\delta} \quad (F.30)$$

Expression (F.28) is recast as

$$z^{\delta 1} = \left\{ e^{\gamma_{rs}^{\delta} z^+} \left(\gamma_{rs}^{\delta} \cos\left(\frac{q\pi z^+}{c}\right) + \frac{q\pi}{c} \sin\left(\frac{q\pi z^+}{c}\right) \right) - \gamma_{rs}^{\delta} \right\} \frac{e^{-\gamma_{rs}^{\delta} L_3^+}}{\left(\frac{q\pi}{c}\right)^2 + (\gamma_{rs}^{\delta})^2} \quad (\text{F.31})$$

which becomes

$$z^{\delta 1} = \left\{ 2\gamma_{rs}^{\delta} e^{\frac{1}{2}\gamma_{rs}^{\delta} z^+} \sinh\left(\frac{\gamma_{rs}^{\delta} z^+}{2}\right) + e^{\gamma_{rs}^{\delta} z^+} \cdot \left(\frac{q\pi}{c} \sin\left(\frac{q\pi z^+}{c}\right) - 2\gamma_{rs}^{\delta} \sin^2\left(\frac{q\pi z^+}{2c}\right) \right) \right\} \frac{e^{-\gamma_{rs}^{\delta} L_3^+}}{\left(\frac{q\pi}{c}\right)^2 + (\gamma_{rs}^{\delta})^2} \quad (\text{F.32})$$

Substituting (F.27), (F.29), and (F.30) into (F.19) and using [5, formula 408.16], we find that

$$z^{\delta 2} = \left\{ \gamma_{rs}^{\delta} \cos\left(\frac{q\pi z^+}{c}\right) \sinh\left(\gamma_{rs}^{\delta} (L_3^+ - z^+)\right) - \frac{q\pi}{c} \sin\left(\frac{q\pi z^+}{c}\right) \cosh\left(\gamma_{rs}^{\delta} (L_3^+ - z^+)\right) - (-1)^q \gamma_{rs}^{\delta} \sinh\left(\gamma_{rs}^{\delta} (L_3^+ - c)\right) \right\} / \left\{ \left(\frac{q\pi}{c}\right)^2 + (\gamma_{rs}^{\delta})^2 \right\} \quad (\text{F.33})$$

If both $\frac{q\pi}{c}$ and γ_{rs}^{δ} are zero, then the right-hand sides of (F.32) and (F.33) are to be replaced by their limits as γ_{rs}^{δ} approaches zero while q is held at zero. If γ_{rs}^{δ} is purely real, expressions (F.32) and (F.33) are suitable for calculating (4.74), (4.75), (4.89), and (4.90) at particular values of z because (F.32) and (F.33) are not subject to excessive roundoff error when q is zero and γ_{rs}^{δ} is small.

Substituting (F.21) and (F.22) into (F.15) and using (E.21), we arrive at

$$z^{TE3} = \frac{j e^{-j\beta_{rs}^{TE} L_3^+}}{2} \left\{ \frac{\sin(q^{TE-} z^+) - 2j \sin^2\left(\frac{q^{TE-} z^+}{2}\right)}{q^{TE-}} - \frac{\sin(q^{TE+} z^+) + 2j \sin^2\left(\frac{q^{TE+} z^+}{2}\right)}{q^{TE+}} \right\} \quad (\text{F.34})$$

Using [5, formulas 401.01 and 401.02] to expand the trigonometric functions in (F.21) and applying (E.21), we obtain

$$z^{TE4} = \left[\left\{ -j \left\{ \sin(q^{TE-} c) - \sin(q^{TE-} z^+) \right\} \cos(\beta_{rs}^{TE} L_3^+) \right. \right.$$

$$\begin{aligned}
& +2j \left\{ \sin^2\left(\frac{q^{TE-}c}{2}\right) - \sin^2\left(\frac{q^{TE-}z^+}{2}\right) \right\} \sin(\beta_{rs}^{TE} L_3^+) \Big\} / (2q^{TE-}) \Big] \\
& + \left[j \left\{ \sin(q^{TE+}c) - \sin(q^{TE+}z^+) \right\} \cos(\beta_{rs}^{TE} L_3^+) \right. \\
& \left. + 2j \left\{ \sin^2\left(\frac{q^{TE+}c}{2}\right) - \sin^2\left(\frac{q^{TE+}z^+}{2}\right) \right\} \sin(\beta_{rs}^{TE} L_3^+) \right\} / (2q^{TE+}) \Big] \quad (F.35)
\end{aligned}$$

If $q^{TE\pm} = 0$, then the right-hand sides of (F.34) and (F.35) are to be replaced by their limits as $q^{TE\pm}$ approaches zero. If β_{rs}^{TE} is purely real, expressions (F.34) and (F.35) are suitable for calculating (4.74), (4.75), (4.89), and (4.90) at particular values of z because (F.34) and (F.35) are not subject to excessive roundoff error when $|q^{TE\pm}|$ is small.

If β_{rs}^{TE} is purely imaginary, we use (F.5) to obtain

$$\beta_{rs}^{TE} = -j\gamma_{rs}^{TE} \quad (F.36)$$

Substituting (F.36) into (F.34) and using (E.21) to dispose of the \sin^2 terms, we arrive at

$$z^{TE3} = \frac{e^{-\gamma_{rs}^{TE} L_3^+}}{2} \left\{ \frac{1 - e^{-jq^{TE-}z^+}}{q^{TE-}} + \frac{1 - e^{jq^{TE+}z^+}}{q^{TE+}} \right\} \quad (F.37)$$

where

$$q^{TE-} = \frac{q\pi}{c} + j\gamma_{rs}^{TE} \quad (F.38)$$

$$q^{TE+} = \frac{q\pi}{c} - j\gamma_{rs}^{TE} \quad (F.39)$$

Expression (F.37) is recast as

$$\begin{aligned}
z^{TE3} = & \left\{ e^{\gamma_{rs}^{TE} z^+} \left(\gamma_{rs}^{TE} \sin\left(\frac{q\pi z^+}{c}\right) - \frac{q\pi}{c} \cos\left(\frac{q\pi z^+}{c}\right) \right) \right. \\
& \left. + \frac{q\pi}{c} \right\} \frac{e^{-\gamma_{rs}^{TE} L_3^+}}{(\frac{q\pi}{c})^2 + (\gamma_{rs}^{TE})^2} \quad (F.40)
\end{aligned}$$

which, in view of (E.21), becomes

$$\begin{aligned}
z^{TE3} = & \left\{ e^{\gamma_{rs}^{TE} z^+} \left(\gamma_{rs}^{TE} \sin\left(\frac{q\pi z^+}{c}\right) + \frac{2q\pi}{c} \sin^2\left(\frac{q\pi z^+}{2c}\right) \right) \right. \\
& \left. - \frac{2q\pi}{c} e^{\frac{1}{2}\gamma_{rs}^{TE} z^+} \sinh\left(\frac{\gamma_{rs}^{TE} z^+}{2}\right) \right\} \frac{e^{-\gamma_{rs}^{TE} L_3^+}}{(\frac{q\pi}{c})^2 + (\gamma_{rs}^{TE})^2} \quad (F.41)
\end{aligned}$$

Substituting (F.36), (F.38), and (F.39) into (F.21) and using [5, formula 408.16], we find that

$$z^{TE4} = \left\{ \gamma_{rs}^{TE} \sin\left(\frac{q\pi z^+}{c}\right) \cosh(\gamma_{rs}^{TE}(L_3^+ - z^+)) + \frac{q\pi}{c} \cos\left(\frac{q\pi z^+}{c}\right) \sinh(\gamma_{rs}^{TE}(L_3^+ - z^+)) - \frac{q\pi}{c} (-1)^q \sinh(\gamma_{rs}^{TE}(L_3^+ - c)) \right\} / \left\{ \left(\frac{q\pi}{c}\right)^2 + (\gamma_{rs}^{TE})^2 \right\} \quad (F.42)$$

If $q = 0$, then, as evident from (F.3) and (F.4), the right-hand sides of (F.41) and (F.42) are to be set equal to zero, regardless of the value of γ_{rs}^{TE} . With this reservation, expressions (F.41) and (F.42) are suitable for calculating (4.74), (4.75), (4.89), and (4.90) when γ_{rs}^{TE} is purely real.

The integrals (4.106)–(4.110) are

$$z^{(1)} = \int_{-\frac{\epsilon}{2}}^{\frac{\epsilon}{2}} \left\{ z^{TM1} \cosh(\gamma_{rs}^{TM}(L_3 - z)) + z^{TM2} e^{-\gamma_{rs}^{TM}(L_3 - z)} \right\} \cos\left(\frac{n\pi z^+}{c}\right) dz \quad (F.43)$$

$$z^{(2)} = \int_{-\frac{\epsilon}{2}}^{\frac{\epsilon}{2}} \left\{ z^{TE1} \cosh(\gamma_{rs}^{TE}(L_3 - z)) + z^{TE2} e^{-\gamma_{rs}^{TE}(L_3 - z)} \right\} \cos\left(\frac{n\pi z^+}{c}\right) dz \quad (F.44)$$

$$z^{(3)} = \int_{-\frac{\epsilon}{2}}^{\frac{\epsilon}{2}} \left\{ -z^{TE1} \sinh(\gamma_{rs}^{TE}(L_3 - z)) + z^{TE2} e^{-\gamma_{rs}^{TE}(L_3 - z)} \right\} \sin\left(\frac{n\pi z^+}{c}\right) dz \quad (F.45)$$

$$z^{(4)} = \int_{-\frac{\epsilon}{2}}^{\frac{\epsilon}{2}} \left\{ z^{TE3} \cosh(\gamma_{rs}^{TE}(L_3 - z)) - z^{TE4} e^{-\gamma_{rs}^{TE}(L_3 - z)} \right\} \cos\left(\frac{n\pi z^+}{c}\right) dz \quad (F.46)$$

$$z^{(5)} = \int_{-\frac{\epsilon}{2}}^{\frac{\epsilon}{2}} \left\{ z^{TE5} - z^{TE3} \sinh(\gamma_{rs}^{TE}(L_3 - z)) - z^{TE4} e^{-\gamma_{rs}^{TE}(L_3 - z)} \right\} \sin\left(\frac{n\pi z^+}{c}\right) dz \quad (F.47)$$

where z^+ and z^{TE5} are given by (F.10) and (4.92), respectively. The quantities z^{TM1} , z^{TM2} , z^{TE1} , z^{TE2} , z^{TE3} , and z^{TE4} are given by (F.1)–(F.4). Changing the variable of integration from z to z^+ , substituting $j\beta_{rs}^{TM}$ and $j\beta_{rs}^{TE}$ for γ_{rs}^{TM} and γ_{rs}^{TE} in accordance with (F.5), and using [5, formulas 654.6 and 654.7], we recast (F.43)–(F.47) as

$$z^{(1)} = \int_0^c \left\{ (z^{TM1} + z^{TM2}) \cos \beta^{TM} - j z^{TM2} \sin \beta^{TM} \right\} \cos\left(\frac{n\pi z^+}{c}\right) dz^+ \quad (F.48)$$

$$z^{(2)} = \int_0^c \left\{ (z^{TE1} + z^{TE2}) \cos \beta^{TE} - j z^{TE2} \sin \beta^{TE} \right\} \cos\left(\frac{n\pi z^+}{c}\right) dz^+ \quad (F.49)$$

$$z^{(3)} = \int_0^c \left\{ z^{TE2} \cos \beta^{TE} - j(z^{TE1} + z^{TE2}) \sin \beta^{TE} \right\} \sin\left(\frac{n\pi z^+}{c}\right) dz^+ \quad (F.50)$$

$$z^{(4)} = \int_0^c \left\{ (z^{TE3} - z^{TE4}) \cos \beta^{TE} + j z^{TE4} \sin \beta^{TE} \right\} \cos\left(\frac{n\pi z^+}{c}\right) dz^+ \quad (F.51)$$

$$z^{(5)} = \int_0^c \left\{ z^{TE5} - z^{TE4} \cos \beta^{TE} - j(z^{TE3} - z^{TE4}) \sin \beta^{TE} \right\} \sin\left(\frac{n\pi z^+}{c}\right) dz^+ \quad (F.52)$$

where

$$\beta^\delta = \beta_{rs}^\delta (L_3^+ - z^+), \quad \delta = TE, TM \quad (F.53)$$

in which L_3^+ is given by (F.11).

Factors appearing in (F.48)–(F.52) are [5, formulas 401.05, 401.06, and 401.07]

$$\sin \beta^\delta \sin\left(\frac{n\pi z^+}{c}\right) = \frac{1}{2} \left\{ \cos(n^{\delta+} z^+ - \beta_{rs}^\delta L_3^+) - \cos(n^{\delta-} z^+ + \beta_{rs}^\delta L_3^+) \right\} \quad (F.54)$$

$$\cos \beta^\delta \sin\left(\frac{n\pi z^+}{c}\right) = \frac{1}{2} \left\{ \sin(n^{\delta+} z^+ - \beta_{rs}^\delta L_3^+) + \sin(n^{\delta-} z^+ + \beta_{rs}^\delta L_3^+) \right\} \quad (F.55)$$

$$\sin \beta^\delta \cos\left(\frac{n\pi z^+}{c}\right) = \frac{1}{2} \left\{ \sin(n^{\delta-} z^+ + \beta_{rs}^\delta L_3^+) - \sin(n^{\delta+} z^+ - \beta_{rs}^\delta L_3^+) \right\} \quad (F.56)$$

$$\cos \beta^\delta \cos\left(\frac{n\pi z^+}{c}\right) = \frac{1}{2} \left\{ \cos(n^{\delta-} z^+ + \beta_{rs}^\delta L_3^+) + \cos(n^{\delta+} z^+ - \beta_{rs}^\delta L_3^+) \right\} \quad (F.57)$$

where

$$n^{\delta-} = \frac{n\pi}{c} - \beta_{rs}^\delta \quad (F.58)$$

$$n^{\delta+} = \frac{n\pi}{c} + \beta_{rs}^\delta \quad (F.59)$$

in which δ is either TE or TM . Substituting (F.19), (F.20), (F.56), and (F.57) into (F.48) and (F.49), we obtain

$$z^{(1)} = z^{TM} \quad (F.60)$$

$$z^{(2)} = z^{TE} \quad (F.61)$$

where

$$\begin{aligned} z^\delta = & \left[\left\{ (\sin(q^{\delta-}c + \beta_{rs}^\delta L_3^+) - je^{-j\beta_{rs}^\delta L_3^+})(z_{c-}^\delta + z_{c+}^\delta) - j(z_{s-}^\delta - z_{s+}^\delta) \right. \right. \\ & \cdot \sin(q^{\delta-}c + \beta_{rs}^\delta L_3^+) + j(z_{s-,s-}^\delta + z_{c-,c-}^\delta) - j(z_{s+,s-}^\delta - z_{c+,c-}^\delta) \left. \right\} / (4q^{\delta-}) \Big] \\ & + \left[\left\{ (\sin(q^{\delta+}c - \beta_{rs}^\delta L_3^+) + je^{-j\beta_{rs}^\delta L_3^+})(z_{c-}^\delta + z_{c+}^\delta) - j(z_{s-}^\delta - z_{s+}^\delta) \right. \right. \\ & \cdot \sin(q^{\delta+}c - \beta_{rs}^\delta L_3^+) + j(z_{s-,s+}^\delta - z_{c-,c+}^\delta) - j(z_{s+,s+}^\delta + z_{c+,c+}^\delta) \left. \right\} / (4q^{\delta+}) \Big] \quad (F.62) \end{aligned}$$

in which

$$z_{s\alpha}^\delta = \int_0^c \sin(n^{\delta\alpha}z^+ - (\alpha 1)\beta_{rs}^\delta L_3^+) dz^+ \quad (F.63)$$

$$z_{c\alpha}^\delta = \int_0^c \cos(n^{\delta\alpha}z^+ - (\alpha 1)\beta_{rs}^\delta L_3^+) dz^+ \quad (F.64)$$

$$z_{s\alpha,\gamma}^\delta = \int_0^c \sin(n^{\delta\alpha}z^+ - (\alpha 1)\beta_{rs}^\delta L_3^+) \sin(q^{\delta\gamma}z^+ - (\gamma 1)\beta_{rs}^\delta L_3^+) dz^+ \quad (F.65)$$

$$z_{c\alpha,\gamma}^\delta = \int_0^c \cos(n^{\delta\alpha}z^+ - (\alpha 1)\beta_{rs}^\delta L_3^+) \cos(q^{\delta\gamma}z^+ - (\gamma 1)\beta_{rs}^\delta L_3^+) dz^+ \quad (F.66)$$

Here, $\alpha 1 = +1$ when $\alpha = +$, and $\alpha 1 = -1$ when $\alpha = -$. The quantity $\gamma 1$ is similarly defined. Substitution of (F.19), (F.20), (F.54), and (F.55) into (F.50) yields

$$\begin{aligned} z^{(3)} = & \left[\left\{ (z_{s+}^{TE} + z_{s-}^{TE}) \sin(q^{TE-}c + \beta_{rs}^{TE} L_3^+) - (j \sin(q^{TE-}c + \beta_{rs}^{TE} L_3^+) \right. \right. \\ & + e^{-j\beta_{rs}^{TE} L_3^+})(z_{c+}^{TE} - z_{c-}^{TE}) - (z_{s+,s-}^{TE} - z_{c+,c-}^{TE}) - (z_{s-,s-}^{TE} + z_{c-,c-}^{TE}) \left. \right\} / (4q^{TE-}) \Big] \\ & + \left[\left\{ (z_{s+}^{TE} + z_{s-}^{TE}) \sin(q^{TE+}c - \beta_{rs}^{TE} L_3^+) + (-j \sin(q^{TE+}c - \beta_{rs}^{TE} L_3^+) \right. \right. \\ & + e^{-j\beta_{rs}^{TE} L_3^+})(z_{c+}^{TE} - z_{c-}^{TE}) - (z_{s+,s+}^{TE} + z_{c+,c+}^{TE}) - (z_{s-,s+}^{TE} - z_{c-,c+}^{TE}) \left. \right\} / (4q^{TE+}) \Big] \quad (F.67) \end{aligned}$$

Evaluating the integrals in (F.63) and (F.64), we obtain

$$z_{s\alpha}^{\delta} = \frac{\cos(\beta_{rs}^{\delta} L_3^+) - \cos(n^{\delta\alpha} c - (\alpha 1) \beta_{rs}^{\delta} L_3^+)}{n^{\delta\alpha}} \quad (F.68)$$

$$z_{c\alpha}^{\delta} = \frac{\sin(n^{\delta\alpha} c - (\alpha 1) \beta_{rs}^{\delta} L_3^+) + (\alpha 1) \sin(\beta_{rs}^{\delta} L_3^+)}{n^{\delta\alpha}} \quad (F.69)$$

If $n^{\delta\alpha}$ is zero, then the right-hand sides of (F.68) and (F.69) are to be replaced by their limits as $n^{\delta\alpha}$ approaches zero. Application of the integration formula (F.16) to (F.65) gives

$$z_{s\alpha, s\gamma}^{\delta} = \frac{\sin((n^{\delta\alpha} - q^{\delta\gamma})c + (\gamma 1 - \alpha 1) \beta_{rs}^{\delta} L_3^+) - \sin((\gamma 1 - \alpha 1) \beta_{rs}^{\delta} L_3^+)}{2(n^{\delta\alpha} - q^{\delta\gamma})} - \frac{\sin((n^{\delta\alpha} + q^{\delta\gamma})c - (\alpha 1 + \gamma 1) \beta_{rs}^{\delta} L_3^+) + \sin((\alpha 1 + \gamma 1) \beta_{rs}^{\delta} L_3^+)}{2(n^{\delta\alpha} + q^{\delta\gamma})} \quad (F.70)$$

Application of the integration formula (F.18) to (F.66) gives

$$z_{c\alpha, c\gamma}^{\delta} = \frac{\sin((n^{\delta\alpha} - q^{\delta\gamma})c + (\gamma 1 - \alpha 1) \beta_{rs}^{\delta} L_3^+) - \sin((\gamma 1 - \alpha 1) \beta_{rs}^{\delta} L_3^+)}{2(n^{\delta\alpha} - q^{\delta\gamma})} + \frac{\sin((n^{\delta\alpha} + q^{\delta\gamma})c - (\alpha 1 + \gamma 1) \beta_{rs}^{\delta} L_3^+) + \sin((\alpha 1 + \gamma 1) \beta_{rs}^{\delta} L_3^+)}{2(n^{\delta\alpha} + q^{\delta\gamma})} \quad (F.71)$$

The sum of (F.70) and (F.71) is

$$z_{s\alpha, s\gamma}^{\delta} + z_{c\alpha, c\gamma}^{\delta} = \frac{\sin((n^{\delta\alpha} - q^{\delta\gamma})c + (\gamma 1 - \alpha 1) \beta_{rs}^{\delta} L_3^+) - \sin((\gamma 1 - \alpha 1) \beta_{rs}^{\delta} L_3^+)}{(n^{\delta\alpha} - q^{\delta\gamma})} \quad (F.72)$$

The difference between (F.70) and (F.71) is

$$z_{s\alpha, s\gamma}^{\delta} - z_{c\alpha, c\gamma}^{\delta} = -\frac{\sin((n^{\delta\alpha} + q^{\delta\gamma})c - (\alpha 1 + \gamma 1) \beta_{rs}^{\delta} L_3^+) + \sin((\alpha 1 + \gamma 1) \beta_{rs}^{\delta} L_3^+)}{(n^{\delta\alpha} + q^{\delta\gamma})} \quad (F.73)$$

Substitution of (F.68), (F.69), (F.72), and (F.73) into (F.62) and (F.67) gives

$$\begin{aligned}
z^\delta &= \frac{j}{4} e^{-j\beta_{rs}^\delta L_3^+} \\
&\cdot \left\{ \frac{(e^{-jn^{\delta-}c} - 1) \sin(q^{\delta-}c + \beta_{rs}^\delta L_3^+) - \sin(n^{\delta-}c + \beta_{rs}^\delta L_3^+) + \sin(\beta_{rs}^\delta L_3^+)}{n^{\delta-}q^{\delta-}} \right. \\
&+ \frac{(1 - ejn^{\delta+}c) \sin(q^{\delta-}c + \beta_{rs}^\delta L_3^+) - \sin(n^{\delta+}c - \beta_{rs}^\delta L_3^+) - \sin(\beta_{rs}^\delta L_3^+)}{n^{\delta+}q^{\delta-}} \\
&+ \frac{(e^{-jn^{\delta-}c} - 1) \sin(q^{\delta+}c - \beta_{rs}^\delta L_3^+) + \sin(n^{\delta-}c + \beta_{rs}^\delta L_3^+) - \sin(\beta_{rs}^\delta L_3^+)}{n^{\delta-}q^{\delta+}} \\
&+ \left. \frac{(1 - ejn^{\delta+}c) \sin(q^{\delta+}c - \beta_{rs}^\delta L_3^+) + \sin(n^{\delta+}c - \beta_{rs}^\delta L_3^+) + \sin(\beta_{rs}^\delta L_3^+)}{n^{\delta+}q^{\delta+}} \right\} \\
&+ \frac{j}{4} \left\{ \frac{\sin((n^{\delta-} - q^{\delta-})c)}{(n^{\delta-} - q^{\delta-})q^{\delta-}} + \frac{\sin((n^{\delta+} + q^{\delta-})c)}{(n^{\delta+} + q^{\delta-})q^{\delta-}} \right. \\
&- \frac{\sin((n^{\delta-} + q^{\delta+})c)}{(n^{\delta-} + q^{\delta+})q^{\delta+}} - \left. \frac{\sin((n^{\delta+} - q^{\delta+})c)}{(n^{\delta+} - q^{\delta+})q^{\delta+}} \right\} \quad (F.74)
\end{aligned}$$

$$\begin{aligned}
z^{(3)} &= \frac{1}{4} e^{-j\beta_{rs}^{TE} L_3^+} \\
&\cdot \left\{ \frac{(1 - ejn^{TE-}c) \sin(q^{TE-}c + \beta_{rs}^{TE} L_3^+) + \sin(n^{TE-}c + \beta_{rs}^{TE} L_3^+) - \sin(\beta_{rs}^{TE} L_3^+)}{n^{TE-}q^{TE-}} \right. \\
&+ \frac{(1 - ejn^{TE+}c) \sin(q^{TE-}c + \beta_{rs}^{TE} L_3^+) - \sin(n^{TE+}c - \beta_{rs}^{TE} L_3^+) - \sin(\beta_{rs}^{TE} L_3^+)}{n^{TE+}q^{TE-}} \\
&+ \frac{(1 - ejn^{TE-}c) \sin(q^{TE+}c - \beta_{rs}^{TE} L_3^+) - \sin(n^{TE-}c + \beta_{rs}^{TE} L_3^+) + \sin(\beta_{rs}^{TE} L_3^+)}{n^{TE-}q^{TE+}} \\
&+ \left. \frac{(1 - ejn^{TE+}c) \sin(q^{TE+}c - \beta_{rs}^{TE} L_3^+) + \sin(n^{TE+}c - \beta_{rs}^{TE} L_3^+) + \sin(\beta_{rs}^{TE} L_3^+)}{n^{TE+}q^{TE+}} \right\} \\
&+ \frac{1}{4} \left\{ - \frac{\sin((n^{TE-} - q^{TE-})c)}{(n^{TE-} - q^{TE-})q^{TE-}} + \frac{\sin((n^{TE+} + q^{TE-})c)}{(n^{TE+} + q^{TE-})q^{TE-}} \right. \\
&+ \frac{\sin((n^{TE-} + q^{TE+})c)}{(n^{TE-} + q^{TE+})q^{TE+}} - \left. \frac{\sin((n^{TE+} - q^{TE+})c)}{(n^{TE+} - q^{TE+})q^{TE+}} \right\} \quad (F.75)
\end{aligned}$$

The term $\{\sin((n^{\delta-} - q^{\delta-})c)\}/\{(n^{\delta-} - q^{\delta-})q^{\delta-}\}$ in (F.74) becomes infinite as $q^{\delta-}$ approaches zero. We subtract $\{\sin(n^{\delta-}c)\}/(n^{\delta-}q^{\delta-})$ from this term to render it finite as $q^{\delta-}$ approaches zero. To maintain equality in (F.74), we add an appropriate quantity to the term whose denominator is $n^{\delta-}q^{\delta-}$. Similarly treating the ill-behaved terms whose denominators are $(n^{\delta+} + q^{\delta-})q^{\delta-}$, $(n^{\delta-} + q^{\delta+})q^{\delta+}$, and $(n^{\delta+} - q^{\delta+})q^{\delta+}$ in (F.74), we arrive, with the help of [5, formulas 401.01 and 401.02], at

$$z^{\delta} = \frac{j}{4}(D^{\delta}G^{\delta} + c^2F^{\delta}) \quad (\text{F.76})$$

where

$$D^{\delta} = \left(\frac{e^{-jn^{\delta-}c} - 1}{n^{\delta-}} + \frac{1 - e^{jn^{\delta+}c}}{n^{\delta+}} \right) e^{-j\beta_{rs}^{\delta}L_3^+} \quad (\text{F.77})$$

$$G^{\delta} = \frac{\sin(q^{\delta-}c + \beta_{rs}^{\delta}L_3^+) - \sin(\beta_{rs}^{\delta}L_3^+)}{q^{\delta-}} + \frac{\sin(q^{\delta+}c - \beta_{rs}^{\delta}L_3^+) + \sin(\beta_{rs}^{\delta}L_3^+)}{q^{\delta+}} \quad (\text{F.78})$$

$$F^{\delta} = -f(n^{\delta-}c, -q^{\delta-}c) + f(n^{\delta+}c, q^{\delta-}c) - f(n^{\delta-}c, q^{\delta+}c) + f(n^{\delta+}c, -q^{\delta+}c) \quad (\text{F.79})$$

in which

$$f(x, y) = \frac{1}{y} \left(\frac{\sin(x+y)}{x+y} - \frac{\sin x}{x} \right) \quad (\text{F.80})$$

If we divided the right-hand side of (F.74) by j , changed the signs of the terms whose denominators are $n^{\delta-}q^{\delta-}$, $(n^{\delta-} - q^{\delta-})q^{\delta-}$, $n^{\delta-}q^{\delta+}$, and $(n^{\delta-} + q^{\delta+})q^{\delta+}$, and then replaced δ by TE , we would have the right-hand side of (F.75). Therefore, we can, from inspection of (F.76)–(F.80), write

$$z^{(3)} = \frac{1}{4}(D^{(3)}G^{(3)} + c^2F^{(3)}) \quad (\text{F.81})$$

where

$$D^{(3)} = \left(\frac{1 - e^{-jn^{TE-}c}}{n^{TE-}} + \frac{1 - e^{jn^{TE+}c}}{n^{TE+}} \right) e^{-j\beta_{rs}^{TE}L_3^+} \quad (\text{F.82})$$

$$G^{(3)} = G^{TE} \quad (\text{F.83})$$

$$F^{(3)} = f(n^{TE-}c, -q^{TE-}c) + f(n^{TE+}c, q^{TE-}c) \\ + f(n^{TE-}c, q^{TE+}c) + f(n^{TE+}c, -q^{TE+}c) \quad (F.84)$$

In (F.83), G^{TE} is given by (F.78) with δ replaced by TE .

In this paragraph and the next two paragraphs, we obtain formulas that are, when β_{rs}^δ is purely real, suitable for calculating D^δ , $D^{(3)}$, G^δ , and the f 's in (F.79) and (F.84). Concerned about roundoff error when $|n^{\delta\pm}|$ is small, we use (E.21) to express D^δ of (F.77) and $D^{(3)}$ of (F.82) as

$$D^\delta = \left\{ \frac{-j \sin(n^{\delta-}c) - 2 \sin^2(\frac{n^{\delta-}c}{2})}{n^{\delta-}} + \frac{-j \sin(n^{\delta+}c) + 2 \sin^2(\frac{n^{\delta+}c}{2})}{n^{\delta+}} \right\} e^{-j\beta_{rs}^\delta L_3^+} \quad (F.85)$$

$$D^{(3)} = \left\{ \frac{j \sin(n^{TE-}c) + 2 \sin^2(\frac{n^{TE-}c}{2})}{n^{TE-}} + \frac{-j \sin(n^{TE+}c) + 2 \sin^2(\frac{n^{TE+}c}{2})}{n^{TE+}} \right\} \\ \cdot e^{-j\beta_{rs}^{TE} L_3^+} \quad (F.86)$$

If $n^{\delta\pm} = 0$, then the $1/n^{\delta\pm}$ terms in (F.85) and (F.86) must be replaced by their limits as $n^{\delta\pm}$ approaches zero. The right-hand sides of (F.85) and (F.86) were purposely expressed so that the values of these limits are obvious.

Concerned about roundoff error when $|q^{\delta\pm}|$ is small, we use [5, formulas 401.01 and 401.02] and (E.21) to express G^δ of (F.78) as

$$G^\delta = \frac{\sin(q^{\delta-}c) \cos(\beta_{rs}^\delta L_3^+) - 2 \sin^2(\frac{q^{\delta-}c}{2}) \sin(\beta_{rs}^\delta L_3^+)}{q^{\delta-}} \\ + \frac{\sin(q^{\delta+}c) \cos(\beta_{rs}^\delta L_3^+) + 2 \sin^2(\frac{q^{\delta+}c}{2}) \sin(\beta_{rs}^\delta L_3^+)}{q^{\delta+}} \quad (F.87)$$

If $q^{\delta\pm} = 0$, then the right-hand side of (F.87) must be replaced by its limit as $q^{\delta\pm}$ approaches zero.

Care must also be taken to avoid excessive roundoff error in the calculation of $f(x, y)$ of (F.80). From (F.23), (F.24), (F.58), and (F.59), we have

$$(n^{\delta-} - q^{\delta-})c = (n - q)\pi \quad (F.88)$$

$$(n^{\delta+} + q^{\delta-})c = (n + q)\pi \quad (F.89)$$

$$(n^{\delta-} + q^{\delta+})c = (n + q)\pi \quad (\text{F.90})$$

$$(n^{\delta+} - q^{\delta+})c = (n - q)\pi \quad (\text{F.91})$$

so that, for every f that appears in (F.79) and (F.84),

$$x + y = (n \pm q)\pi \quad (\text{F.92})$$

Equation (F.92) reduces (F.80) to

$$f(x, y) = -\frac{\sin x}{yx}, \quad x + y \neq 0 \quad (\text{F.93})$$

$$f(x, y) = \frac{y - \sin y}{y^2}, \quad x + y = 0 \quad (\text{F.94})$$

Expression (F.94) was obtained by taking the limit of the right-hand side of (F.80) as $x + y$ approaches zero. If $|y| \leq \frac{\pi}{2}$, we use (F.92) to recast (F.93) as

$$f(x, y) = \frac{(-1)^{n \pm q} \sin y}{yx}, \quad \begin{cases} x + y \neq 0 \\ |y| \leq \frac{\pi}{2} \end{cases} \quad (\text{F.95})$$

If $|y| \leq 0.1$, we replace the right-hand side of (F.94) by the series approximation [5, formula 415.01]

$$f(x, y) = \frac{y}{3!} - \frac{y^3}{5!} + \frac{y^5}{7!}, \quad \begin{cases} x + y = 0 \\ |y| \leq 0.1 \end{cases} \quad (\text{F.96})$$

Collecting the results (F.93)–(F.96), we have

$$f(x, y) = \begin{cases} -\frac{\sin x}{yx}, & \begin{cases} x + y \neq 0 \\ |y| > \frac{\pi}{2} \end{cases} \\ \frac{(-1)^{n \pm q} \sin y}{yx}, & \begin{cases} x + y \neq 0 \\ |y| \leq \frac{\pi}{2} \end{cases} \\ \frac{y - \sin y}{y^2}, & \begin{cases} x + y = 0 \\ |y| > 0.1 \end{cases} \\ \frac{y}{3!} - \frac{y^3}{5!} + \frac{y^5}{7!}, & \begin{cases} x + y = 0 \\ |y| \leq 0.1 \end{cases} \end{cases} \quad (\text{F.97})$$

where $(n \pm q)$ is the integer that satisfies (F.92).

In this paragraph and the next paragraph, we obtain formulas that are, when β_{rs}^δ is purely imaginary, suitable for calculating D^δ , $D^{(3)}$, G^δ , F^δ , and $F^{(3)}$. We arrive at these formulas by substituting (F.27), (F.29), (F.30), and

$$n^{\delta-} = \frac{n\pi}{c} + j\gamma_{rs}^\delta \quad (\text{F.98})$$

$$n^{\delta+} = \frac{n\pi}{c} - j\gamma_{rs}^\delta \quad (\text{F.99})$$

into (F.77), (F.82), (F.78), (F.79), and (F.84). Substitution of (F.27), (F.98), and (F.99) into (F.77) gives

$$D^\delta = -\frac{4j\gamma_{rs}^\delta c^2 e^{-\gamma_{rs}^\delta(L_3^+ - \frac{c}{2})} \sinh(\frac{\gamma_{rs}^\delta c}{2})}{(n\pi)^2 + (\gamma_{rs}^\delta c)^2}, \quad n \text{ even} \quad (\text{F.100})$$

$$D^\delta = \frac{4j\gamma_{rs}^\delta c^2 e^{-\gamma_{rs}^\delta(L_3^+ - \frac{c}{2})} \cosh(\frac{\gamma_{rs}^\delta c}{2})}{(n\pi)^2 + (\gamma_{rs}^\delta c)^2}, \quad n \text{ odd} \quad (\text{F.101})$$

If both n and γ_{rs}^δ are zero, then the right-hand side of (F.100) must be replaced by its limit as γ_{rs}^δ approaches zero while n is held at zero. This limit is $-2jc$. Substitution of (F.27), (F.98), and (F.99) into (F.82) gives

$$D^{(3)} = -\frac{4n\pi c e^{-\gamma_{rs}^{TE}(L_3^+ - \frac{c}{2})} \sinh(\frac{\gamma_{rs}^{TE} c}{2})}{(n\pi)^2 + (\gamma_{rs}^{TE} c)^2}, \quad n \text{ even} \quad (\text{F.102})$$

$$D^{(3)} = \frac{4n\pi c e^{-\gamma_{rs}^{TE}(L_3^+ - \frac{c}{2})} \cosh(\frac{\gamma_{rs}^{TE} c}{2})}{(n\pi)^2 + (\gamma_{rs}^{TE} c)^2}, \quad n \text{ odd} \quad (\text{F.103})$$

If both n and γ_{rs}^{TE} are zero, then the right-hand side of (F.102) must be replaced by its limit as γ_{rs}^{TE} approaches zero while n is held at zero. This limit is zero. Substitution of (F.27), (F.29), and (F.30) into (F.78) and use of [5, formula 408.16] lead to

$$G^\delta = \frac{2\gamma_{rs}^\delta c^2 \{ \sinh(\gamma_{rs}^\delta L_3^+) - (-1)^q \sinh(\gamma_{rs}^\delta (L_3^+ - c)) \}}{(q\pi)^2 + (\gamma_{rs}^\delta c)^2} \quad (\text{F.104})$$

If both q and γ_{rs}^δ are zero, then the right-hand side of (F.104) must be replaced by its limit as γ_{rs}^δ approaches zero while q is held at zero. This limit is $2c$.

Substituting (F.80), (F.29), (F.30), (F.98), and (F.99) into (F.79), using [5, formula 408.16], and assuming that n is precisely an integer, we obtain

$$F^\delta = -\frac{2j\gamma_{rs}^\delta c}{(q\pi)^2 + (\gamma_{rs}^\delta c)^2} \left\{ \frac{\sin((n+q)\pi)}{(n+q)\pi} + \frac{\sin((n-q)\pi)}{(n-q)\pi} - \frac{2\gamma_{rs}^\delta c(-1)^n \sinh(\gamma_{rs}^\delta c)}{(n\pi)^2 + (\gamma_{rs}^\delta c)^2} \right\} \quad (F.105)$$

Similarly, (F.84) yields

$$F^{(3)} = \frac{2\pi}{(q\pi)^2 + (\gamma_{rs}^{TE} c)^2} \left\{ q \left(\frac{\sin((n+q)\pi)}{(n+q)\pi} - \frac{\sin((n-q)\pi)}{(n-q)\pi} \right) + \frac{2n\gamma_{rs}^{TE} c(-1)^n \sinh(\gamma_{rs}^{TE} c)}{(n\pi)^2 + (\gamma_{rs}^{TE} c)^2} \right\} \quad (F.106)$$

If $q = \pm n$, then the right-hand sides of (F.105) and (F.106) must be replaced by their limits as q approaches $\pm n$ while n is held at its integer value. Thus, assuming that both q and n are non-negative integers, forms suitable for calculation are:

$$F^\delta = \frac{4j(\gamma_{rs}^\delta c)^2(-1)^n \sinh(\gamma_{rs}^\delta c)}{\{(q\pi)^2 + (\gamma_{rs}^\delta c)^2\} \{(n\pi)^2 + (\gamma_{rs}^\delta c)^2\}}, \quad q \neq n \quad (F.107)$$

$$F^\delta = \frac{2j\gamma_{rs}^\delta c}{(q\pi)^2 + (\gamma_{rs}^\delta c)^2} \left\{ \frac{2\gamma_{rs}^\delta c(-1)^n \sinh(\gamma_{rs}^\delta c)}{(n\pi)^2 + (\gamma_{rs}^\delta c)^2} - 1 \right\}, \quad q = n \neq 0 \quad (F.108)$$

$$F^\delta = -\frac{4j \left\{ \gamma_{rs}^\delta c - \sinh(\gamma_{rs}^\delta c) \right\}}{(\gamma_{rs}^\delta c)^2}, \quad \begin{cases} q = n = 0 \\ \gamma_{rs}^\delta c > 0.1 \end{cases} \quad (F.109)$$

$$F^\delta = 4j \left\{ \frac{\gamma_{rs}^\delta c}{3!} + \frac{(\gamma_{rs}^\delta c)^3}{5!} + \frac{(\gamma_{rs}^\delta c)^5}{7!} \right\}, \quad \begin{cases} q = n = 0 \\ \gamma_{rs}^\delta c \leq 0.1 \end{cases} \quad (F.110)$$

$$F^{(3)} = \frac{4\pi n \gamma_{rs}^{TE} c(-1)^n \sinh(\gamma_{rs}^{TE} c)}{\{(q\pi)^2 + (\gamma_{rs}^{TE} c)^2\} \{(n\pi)^2 + (\gamma_{rs}^{TE} c)^2\}}, \quad q \neq n \quad (F.111)$$

$$F^{(3)} = \frac{2\pi n}{(q\pi)^2 + (\gamma_{rs}^{TE} c)^2} \left\{ \frac{2\gamma_{rs}^{TE} c(-1)^n \sinh(\gamma_{rs}^{TE} c)}{(n\pi)^2 + (\gamma_{rs}^{TE} c)^2} - 1 \right\}, \quad q = n \neq 0 \quad (F.112)$$

$$F^{(3)} = 0, \quad q = n = 0 \quad (F.113)$$

If $q = 0$ or $n = 0$ and if $\gamma_{rs}^\delta = 0$, then (F.107) and (F.111) must be replaced by their limits as γ_{rs}^δ approaches zero. The series expansion [5, formula 657.1] was used to obtain (F.110).

Comparing (F.21) with (F.19), we note that

$$z^{TE4} = -j\tilde{z}^{TE2} \quad (F.114)$$

where \tilde{z}^{TE2} is the right-hand side of (F.21) with the sign of the $1/q^{\delta+}$ term changed and with δ replaced by TE . Comparing (F.22) with (F.20), we find that

$$z^{TE3} - z^{TE4} = j(\tilde{z}^{TE1} + \tilde{z}^{TE2}) \quad (F.115)$$

where $(\tilde{z}^{TE1} + \tilde{z}^{TE2})$ is the right-hand side of (F.20) with the sign of the $1/q^{\delta+}$ term changed and with δ replaced by TE . Substituting (F.114) and (F.115) into (F.51) and (F.52) and comparing with (F.49) and (F.50), we obtain

$$z^{(4)} = j\tilde{z}^2 \quad (F.116)$$

$$z^{(5)} = j\tilde{z}^{(3)} + \int_0^c z^{TE5} \sin\left(\frac{n\pi z^+}{c}\right) dz^+ \quad (F.117)$$

where $\tilde{z}^{(2)}$ is the right-hand side of (F.49) with the signs of the $1/q^{TE+}$ terms changed. Similarly, $\tilde{z}^{(3)}$ is the right-hand side of (F.50) with the signs of the $1/q^{TE+}$ terms changed. According to (F.61), $z^{(2)}$ is given by the right-hand side of (F.76) with δ replaced by TE .

Substituting (F.76) into (F.116) and noting that the integrations that were performed in obtaining (F.76) from (F.49) did not introduce any $1/q^{\delta+}$ factors in addition to those in (F.19) and (F.20), we obtain

$$z^{(4)} = -\frac{1}{4}(D^{TE}G^{(4)} + c^2F^{(4)}) \quad (F.118)$$

where D^{TE} is given by (F.77) with δ replaced by TE , $G^{(4)}$ is given by the right-hand side of (F.78) with the sign of the $1/q^{\delta+}$ term changed and with δ replaced by TE , and $F^{(4)}$ is given by the right-hand side of (F.79) with the signs of the $1/q^{\delta+}$ terms changed and with δ replaced by TE . Substituting (4.92) and (F.81) into (F.117), noting that the integrations that were performed in obtaining (F.81) from (F.50) did not introduce any $1/q^{TE+}$ factors in addition to those in (F.19) and (F.20), and using (F.5), we obtain

$$z^{(5)} = \frac{j\beta_{rs}^{TE} z_{ss}^{TE}}{(k_{rs}^{TE})^2} + \frac{j}{4}(D^{(3)}G^{(4)} + c^2F^{(5)}) \quad (F.119)$$

where $D^{(3)}$ is given by (F.82), $G^{(4)}$ is the same as in (F.118), and $F^{(5)}$ is given by the right-hand side of (F.84) with the signs of the $1/q^{TE+}$ terms changed. Moreover,

$$z_{ss}^{TE} = \int_0^c \sin\left(\frac{q\pi z^+}{c}\right) \sin\left(\frac{n\pi z^+}{c}\right) dz^+ \quad (F.120)$$

Assuming that both n and q are non-negative integers, we have

$$z_{ss}^{TE} = \begin{cases} \frac{c}{2}, & n = q \neq 0 \\ 0, & \text{otherwise} \end{cases} \quad (F.121)$$

If β_{rs}^{TE} is purely real, then D^{TE} and $D^{(3)}$ are suitably given by (F.85) and (F.86). Assuming that β_{rs}^{TE} is purely real, we proceed to obtain suitable expressions for $G^{(4)}$, $F^{(4)}$, and $F^{(5)}$. From (F.87), we have

$$G^{(4)} = \frac{\sin(q^{TE-}c) \cos(\beta_{rs}^{TE} L_3^+) - 2 \sin^2\left(\frac{q^{TE-}c}{2}\right) \sin(\beta_{rs}^{TE} L_3^+)}{q^{TE-}} - \frac{\sin(q^{TE+}c) \cos(\beta_{rs}^{TE} L_3^+) + 2 \sin^2\left(\frac{q^{TE+}c}{2}\right) \sin(\beta_{rs}^{TE} L_3^+)}{q^{TE+}} \quad (F.122)$$

From (F.79) and (F.84), we have

$$F^{(4)} = -f(n^{TE-}c, -q^{TE-}c) + f(n^{TE+}c, q^{TE-}c) + f(n^{TE-}c, q^{TE+}c) - f(n^{TE+}c, -q^{TE+}c) \quad (F.123)$$

$$F^{(5)} = f(n^{TE-}c, -q^{TE-}c) + f(n^{TE+}c, q^{TE-}c) - f(n^{TE-}c, q^{TE+}c) - f(n^{TE+}c, -q^{TE+}c) \quad (F.124)$$

where $f(x, y)$ is given by (F.97).

If β_{rs}^{TE} is purely imaginary, we point out that D^{TE} is suitably given by (F.100) and (F.101) with δ replaced by TE and that $D^{(3)}$ is suitably given by (F.102) and (F.103). Assuming that β_{rs}^{TE} is purely imaginary, we proceed to obtain a suitable expression for $G^{(4)}$. From (F.78), we have

$$G^{(4)} = \frac{\sin(q^{TE-}c + \beta_{rs}^{TE} L_3^+) - \sin(\beta_{rs}^{TE} L_3^+)}{q^{TE-}} - \frac{\sin(q^{TE+}c - \beta_{rs}^{TE} L_3^+) + \sin(\beta_{rs}^{TE} L_3^+)}{q^{TE+}} \quad (F.125)$$

Substitution of (F.27), (F.29), and (F.30) into (F.125) leads to

$$G^{(4)} = \frac{j2\pi qc \left\{ \sinh(\gamma_{rs}^{TE} L_3^+) - (-1)^q \sinh(\gamma_{rs}^{TE} (L_3^+ - c)) \right\}}{(q\pi)^2 + (\gamma_{rs}^{TE} c)^2} \quad (F.126)$$

If both q and γ_{rs}^{TE} are zero, then the right-hand side of (F.126) must be replaced by its limit as γ_{rs}^{TE} approaches zero while q is held at zero. This limit is zero.

In this paragraph, formulas are obtained for calculating $F^{(4)}$ and $F^{(5)}$ when β_{rs}^{TE} is purely imaginary. Substituting (F.29), (F.30), (F.98), and (F.99) into (F.123), using [5, formula 408.16], and assuming that n is precisely an integer, we obtain

$$F^{(4)} = \frac{2\pi q}{(q\pi)^2 + (\gamma_{rs}^{TE} c)^2} \left\{ \frac{\sin((n+q)\pi)}{(n+q)\pi} + \frac{\sin((n-q)\pi)}{(n-q)\pi} - \frac{2\gamma_{rs}^{TE} c (-1)^n \sinh(\gamma_{rs}^{TE} c)}{(n\pi)^2 + (\gamma_{rs}^{TE} c)^2} \right\} \quad (F.127)$$

Similarly, (F.124) yields

$$F^{(5)} = \frac{2j}{(q\pi)^2 + (\gamma_{rs}^{TE} c)^2} \left\{ -\gamma_{rs}^{TE} c \left(\frac{\sin((n+q)\pi)}{(n+q)\pi} - \frac{\sin((n-q)\pi)}{(n-q)\pi} \right) + \frac{2\pi^2 n q (-1)^n \sinh(\gamma_{rs}^{TE} c)}{(n\pi)^2 + (\gamma_{rs}^{TE} c)^2} \right\} \quad (F.128)$$

If $q = \pm n$, then the right-hand sides of (F.127) and (F.128) must be replaced by their limits as q approaches $\pm n$ while n is held at its integer value. Thus, assuming that both q and n are non-negative integers, forms suitable for calculation are:

$$F^{(4)} = -\frac{4\pi q \gamma_{rs}^{TE} c (-1)^n \sinh(\gamma_{rs}^{TE} c)}{\{(q\pi)^2 + (\gamma_{rs}^{TE} c)^2\} \{(n\pi)^2 + (\gamma_{rs}^{TE} c)^2\}}, \quad q \neq n \quad (F.129)$$

$$F^{(4)} = \frac{2\pi q}{(q\pi)^2 + (\gamma_{rs}^{TE} c)^2} \left\{ 1 - \frac{2\gamma_{rs}^{TE} c (-1)^n \sinh(\gamma_{rs}^{TE} c)}{(n\pi)^2 + (\gamma_{rs}^{TE} c)^2} \right\}, \quad q = n \neq 0 \quad (F.130)$$

$$F^{(4)} = 0, \quad q = n = 0 \quad (F.131)$$

$$F^{(5)} = \frac{4j\pi^2 n q (-1)^n \sinh(\gamma_{rs}^{TE} c)}{(q\pi)^2 + (\gamma_{rs}^{TE} c)^2}, \quad q \neq n \quad (F.132)$$

$$F^{(5)} = \frac{2j}{(q\pi)^2 + (\gamma_{rs}^{TE}c)^2} \left\{ \gamma_{rs}^{TE}c + \frac{2(n\pi)^2(-1)^n \sinh(\gamma_{rs}^{TE}c)}{(n\pi)^2 + (\gamma_{rs}^{TE}c)^2} \right\}, \quad q = n \neq 0 \quad (\text{F.133})$$

$$F^{(5)} = 0, \quad q = n = 0 \quad (\text{F.134})$$

If $q = 0$ or $n = 0$ and if $\gamma_{rs}^{TE} = 0$, then the right-hand side of (F.129) must be replaced by its limit as γ_{rs}^{TE} approaches zero. If $n = 0$ and $\gamma_{rs}^{TE} = 0$, then the right-hand side of (F.132) must be replaced by zero.

The results obtained in Appendix F are cataloged in Table F.1. In Table F.1, the quantity in the first column is given by the equation whose number appears in the second column when the nature of either γ_{rs}^{TM} or γ_{rs}^{TE} is indicated in the third column.

Table F.1: Results in Appendix F

Quantity	Equation Number	Circumstance
z^{TM1}	(F.25)	γ_{rs}^{TM} purely imaginary
z^{TM2}	(F.26)	γ_{rs}^{TM} purely imaginary
z^{TM1}	(F.32)	γ_{rs}^{TM} purely real
z^{TM2}	(F.33)	γ_{rs}^{TM} purely real
z^{TE1}	(F.25)	γ_{rs}^{TE} purely imaginary
z^{TE2}	(F.26)	γ_{rs}^{TE} purely imaginary
z^{TE3}	(F.34)	γ_{rs}^{TE} purely imaginary
z^{TE4}	(F.35)	γ_{rs}^{TE} purely imaginary
z^{TE1}	(F.32)	γ_{rs}^{TE} purely real
z^{TE2}	(F.33)	γ_{rs}^{TE} purely real
z^{TE3}	(F.41)	γ_{rs}^{TE} purely real
z^{TE4}	(F.42)	γ_{rs}^{TE} purely real
$z^{(1)}$	(F.60),(F.76),(F.85),(F.87), (F.79),(F.97)	γ_{rs}^{TM} purely imaginary
$z^{(1)}$	(F.60),(F.76),(F.100),(F.101), (F.104),(F.107)–(F.110)	γ_{rs}^{TM} purely real
$z^{(2)}$	(F.60),(F.76),(F.85),(F.87), (F.79),(F.97)	γ_{rs}^{TE} purely imaginary
$z^{(3)}$	(F.81),(F.86),(F.83), (F.87),(F.84),(F.97)	γ_{rs}^{TE} purely imaginary
$z^{(4)}$	(F.118),(F.85),(F.122), (F.123),(F.97)	γ_{rs}^{TE} purely imaginary
$z^{(5)}$	(F.119),(F.121),(F.86), (F.122),(F.124),(F.97)	γ_{rs}^{TE} purely imaginary
$z^{(2)}$	(F.60),(F.76),(F.100),(F.101), (F.104),(F.107)–(F.110)	γ_{rs}^{TE} purely real
$z^{(3)}$	(F.81),(F.102),(F.103),(F.83) (F.104),(F.111)–(F.113)	γ_{rs}^{TE} purely real
$z^{(4)}$	(F.118),(F.100),(F.101), (F.126),(F.129)–(F.131)	γ_{rs}^{TE} purely real
$z^{(5)}$	(F.119),(F.121),(F.102), (F.103),(F.126),(F.132)–(F.134)	γ_{rs}^{TE} purely real

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